A THREE-VALUED LOGIC

A Thesis

Presented to

The Faculty of the Department of Mathematics

Sam Houston State University

In Partial Fulfillment

of the Requirements for the Degree of

Master of Arts

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by

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August, 1973

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A THESIS

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ABSTRACT

Ermis, Alfred J., A Three-Valued Logic. Master of Arts (Mathematics), August, 1973, Sam Houston State University, Huntsville, Texas.

The logic used in mathematics today asserts that any statement is either true or false. It is the function of logic to establish a way of determining the truth or falsity of a given statement. However, the truth or falsity of some statements either has not been, or can not be determined in the classical logic. Thus, these statements cannot be used in logical arguments since their truth value has not been established. Thus the question arises, "Could a three-valued logic be developed in which the statements mentioned above would receive a third value, say M for maybe?"

Several three-valued systems of logic have been developed. The basic truth tables used for the development in this paper were defined in 1938 by S. C. Kleene. However, Kleene adheres strictly to the classical definition of what is meant by a formula being a tautology. Thus in his development, the formula $p \rightarrow p$ is not a tautology. The notion of a formula being a tautology has been defined in this paper in such a way that the formula $p \rightarrow p$ is a tautology in the three-valued logic. Thus many formulas which would not be tautologies in Kleene's original system are tautologies in the system developed here.

The concepts of equivalence, substitution, and consequence are defined in the three-valued logic in the same way as they are in the classical logic. Their properties

and relationship with the new notion of tautology in the three-valued logic are compared with those in the classical logic. Many of the properties which hold true in the classical logic are also true or partially true in the three-valued logic.

A topic of interest in the classical logic is that of truth functions which generate all possible functions of two arguments. Two such functions, the stroke ($/$) and the dagger ($\downarrow$) function are defined in this paper in such a way that they generate all the functions of two arguments which contain only the usual connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$. However, they do not generate all the possible truth functions of two arguments in the three-valued logic. Thus, a truth function, the square ($\square$) function was defined in such a way that it would generate all possible truth functions of two arguments in the three-valued logic.

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CHAPTER I

INTRODUCTION

The logic used in mathematics today asserts that any statement is either true or false. It is the function of logic to establish a means by which one can determine whether a given statement is true or false in a given mathematical system. However, a German logician K. Gödel showed that there existed statements which could not be proven or disproven. Also statements such as Fermat's Last Theorem have never been proven or disproven. Thus the truth or falsity of these statements cannot or has not been established. Hence, questions arise as to the universality of two-valued logic. In view of this situation one may ask himself: "Is it possible to develope a three-valued logic, and how would the properties of this logic compare with those of the classical logic?". The logic investigated in this paper is a three-valued logic in the sense that a statement p can be assigned exactly one of three values, T(true), M(maybe), or F(false).

Some preliminary notions must be defined before the development of the three-valued logic can proceed. The connectives from classical logic, $\wedge$ (and), $\vee$ (or), $\neg$ (not), and $\rightarrow$(implies) will be used. A composite sentence will be a declarative sentence in which one or more of the connectives appear. A prime sentence will be a declarative sentence in which none of the above connectives appear, or one which is by choice "indivisible". For example, "3 is a prime number" is a prime sentence, whereas "3 is a prime number and 5 is not a prime number" is a composite sentence. Now a sentence will be a statement if it can be classified as exactly one of true, maybe, or false. Prime statements will be represented by lower case letters such as p, q, r, s, etc. Composite statements will be represented by upper case letters such as P, Q, R, S, etc. Other concepts and symbols will be defined as they are needed in the development of the logic.

Before the truth tables for the above mentioned connectives are defined, it is appropriate that the motivation for the definition be discussed. It is assumed that any statement is either true or false. However, the truth value of a statement cannot always be determined. These kinds of statements will be assigned the value M and will be called "maybe statements". Now, what truth value could be assigned to $p \wedge q$ when p has value T and q has value M? Assuming q either true or false in the classical sense, suppose q is true. In this case, from classical logic, $p \wedge q$ has value T. However, supposing q is false, then $p \wedge q$ would classically have the value F. Thus whenever p has value T and q has value M, $p \wedge q$ is not definitely true nor is it definitely false, thus $p \wedge q$ will be assigned the value M. If however, p has value F and q has value M, then regardless of what value q might assume, $p \wedge q$ will have value F. Similar reasoning is used to motivate the definitions of $p \vee q$, $\neg p$, and $p \rightarrow q$.

The truth values for statements containing only the connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$, are now defined in the form of a truth table below. This truth table has been defined previously by S. C. Kleene in 1938. (1, Rescher, page 34) However, his development of a three-valued logic differs from the development in this paper. This difference will be pointed out later.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T
T	M	F	M	T	M	M
F	M	T	F	M	T	M
M	T	M	M	T	T	M
M	F	M	F	M	M	M
M	M	M	M	M	M	M

Note that for assignments to the prime statements which consist of only the values T or F, the composite statements are assigned the same values they would have in the classical two-valued logic.

Consider now the set of all prime statements. Adjoin to this set the set of statements which can be formed using and possibly repeating and in all possible ways the various connectives mentioned above, and where the truth value for a given assignment to the prime statements is in accordance with the truth table above. The members of the combined sets will be called formulas. The prime statements will be called prime formulas, and the other statements will be called composite formulas. The prime statements which compose a formula will be called prime components. Formulas will be represented symbolically in the same way that prime and composite statements are represented.

Observe that each of the connectives $\wedge$, $\vee$ and $\rightarrow$ above defines a way of associating with each ordered pair of values consisting of T, M, or F, one of the values T, M, or F. Let $W = \{T, M, F\}$. Note that the connective $\wedge$ is a function on $W \times W$ into W. It maps the ordered pairs (T, T) into the element T, (T, F) into F, etc. Similarly, the connectives $\vee$ and $\rightarrow$ are also functions on $W \times W$ into W. Indeed, if P is any composite statement, composed of prime statements $p_1, p_2, \ldots, p_n$, it defines a function from W^n into W no matter what the connectives are. These functions are called truth functions. Clearly, a truth table defines a truth function and conversely, a truth function defines a truth table. Truth functions will be treated in greater detail in Chapter III.

The development of the three-valued logic from this point on, through Chapter II, will closely parallel Robert R. Stoll's development of the classical two-valued logic. (2, Stoll, pages 171-182) As in the classical logic, interest will be centered on formulas which have value T for all assignments of truth values to their prime components. In classical logic, a formula which has value T for all assignments of truth values to the prime components is called a tautology, or alternately, a valid statement. However, in this three-valued logic it is impossible to obtain a formula which will have value T in accordance with the truth table above whenever all the prime components have value M. Observe in the truth table that the last line in the table assigns to each connective the value M. This situation motivates the following definition. A formula will be called a tautology if it is assigned the value T for all assignments of truth values to the prime components, except for the assignments which assign the value M to each of the prime components. If the formula P is a tautology, it will be written $\models P$. Note that if P is a tautology in the three-valued logic it is also a tautology in the classical logic, for if P has value T for all assignments to $p_1, p_2, \ldots, p_n$ except when all are M, then P certainly has value T for those assignments to $p_1, p_2, \ldots, p_n$ which consist of only T's or F's. This definition is the main difference between Kleene's system of logic and the system developed in this paper. Kleene adheres strictly to the classical definition of tautology. (1, Rescher, page 34) Therefore in his system, the statement $p \rightarrow p$ is not a tautology since it is assigned the value M whenever p has value M. However, according to the definition given above, $p \rightarrow p$ is a tautology since $p \rightarrow p$ is assigned the value T whenever p has value T or F.

Some additional tautologies will now be stated. That they are tautologies can easily be verified by means of truth tables. Some of the truth tables will be given as a means of illustration. The reader should refer to the basic truth table defined above as an aid to understanding the truth tables used here. The following formulas are tautologies:

I. $\models (p \wedge q) \rightarrow (p \vee q)$

Proof:

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T
T	M	M	T	T
F	M	F	M	T
M	T	M	T	T
M	F	F	M	T
M	M	M	M	M

II. $\models (p \rightarrow q) \vee (q \rightarrow p)$

Proof:

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \vee (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T
T	M	M	T	T
F	M	T	M	T
M	T	T	M	T
M	F	M	T	T
M	M	M	M	M

III. $\models [p \rightarrow (p \vee q)] \vee [q \rightarrow (p \vee q)]$

Proof: The truth table is left to the reader.

IV. $\models [(p \wedge q) \rightarrow p] \vee [(p \wedge q) \rightarrow q]$

Proof: The truth table is left to the reader.

V. $\models [(p \wedge q) \rightarrow (p \rightarrow q)] \vee [(p \wedge q) \rightarrow (q \rightarrow p)]$

Proof: The truth table is left to the reader.

VI. $\models \left[(p \wedge q) \rightarrow r\right] \vee \left[r \rightarrow (p \vee q)\right]$

Proof:

p	q	r	$(p \wedge q) \rightarrow r$	$r \rightarrow (p \vee q)$	$\left[(p \wedge q) \rightarrow r\right] \vee \left[r \rightarrow (p \vee q)\right]$
T	T	T	T	T	T
T	T	F	F	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T
T	T	M	M	T	T
T	F	M	T	T	T
F	T	M	T	T	T
F	F	M	T	M	T
T	M	T	T	T	T
T	M	F	M	T	T
F	M	T	T	M	T
F	M	F	T	M	T
M	T	T	T	T	T
M	T	F	M	T	T
M	F	T	T	M	T
M	F	F	T	T	T
M	T	M	M	T	T
M	F	M	T	M	T
M	M	T	T	M	T
M	M	F	M	T	T
T	M	M	M	T	T
F	M	M	T	M	T
M	M	M	M	M	M

It has already been said that every tautology in the three-valued logic is also a tautology in the classical logic. It would be interesting to determine what properties, if any, tautologies in the classical logic have in the three-valued logic. There is, in fact, an interesting property which classical tautologies have in the three-valued logic. Before this property of classical tautologies can be presented, some definitions must be made and some properties of the three-valued logic established.

One of the terms which needs to be defined is the notion of two formulas being equivalent. For convenience sake, formulas will be interpreted as truth functions. Let P and Q be any formulas with $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_k$ being their prime components respectively. Now, formula P is equivalent to formula Q (written P eq. Q) if they are equal as truth functions of the list of variables $t_1, t_2, \ldots, t_m$ where t_i appears as a prime component of at least one of P and Q for all $i = 1, 2, \ldots, m$. Observe that P can be expressed as a function of $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_k$ variables simply by treating $q_1, q_2, \ldots, q_k$ as "dummy" variables. Similarly for Q. In terms of truth tables, the definition simply means that if $\{t_1, t_2, \ldots, t_m\}$ is the union of the sets of prime components contained in P and Q respectively, then one could compute the truth tables of P and Q as if both contained $t_1, t_2, \ldots, t_m$ as prime components. Then P eq. Q if the resulting truth tables are the same.

Some of the equivalent formulas which are important in the classical logic are presented here in the setting of the three-valued logic. The truth tables for some will be given, and the others will be left to the reader. The following are equivalent formulas:

I. $p \rightarrow q$ eq. $\neg p \vee q$

Proof:

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T
T	M	M	F	M
F	M	T	T	T
M	T	T	M	T
M	F	M	M	M
M	M	M	M	M

II. $p \rightarrow q$ eq. $\neg q \rightarrow \neg p$

Proof:

p	q	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$	$p \rightarrow q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T
T	M	F	M	M	M
F	M	T	M	T	T
M	T	M	F	T	T
M	F	M	T	M	M
M	M	M	M	M	M

III. $\neg(p \wedge q)$ eq. $\neg p \vee \neg q$

IV. $(p \wedge q)$ eq. $\neg(\neg p \vee \neg q)$

V. $\neg(p \vee q)$ eq. $\neg p \wedge \neg q$

VI. $(p \wedge q) \vee (p \vee q)$ eq. $(p \vee q)$

VII. $(p \leftrightarrow q)$ eq. $(p \rightarrow q) \wedge (q \rightarrow p)$

VIII. $(p \vee q) \vee r$ eq. $p \vee (q \vee r)$

Proof:

p	q	r	$(p \vee q)$	$(p \vee q) \vee r$	$(q \vee r)$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

p	q	r	$(p \vee q)$	$(p \vee q) \vee r$	$(q \vee r)$	$p \vee (q \vee r)$
T	T	M	T	T	T	T
T	F	M	T	T	M	T
F	T	M	T	T	T	T
F	F	M	F	M	M	M
T	M	T	T	T	T	T
T	M	F	T	T	M	T
F	M	T	M	T	T	T
F	M	F	M	M	M	M
M	T	T	T	T	T	T
M	T	F	T	T	T	T
M	F	T	M	T	T	T
M	F	F	M	M	F	M
M	T	M	T	T	T	T
M	F	M	M	M	M	M
M	M	T	M	T	T	T
M	M	F	M	M	M	M
T	M	M	T	T	M	T
F	M	M	M	M	M	M
M	M	M	M	M	M	M

IX. $(p \wedge q) \wedge r$ eq. $p \wedge (q \wedge r)$

X. $p \wedge (q \vee r)$ eq. $(p \wedge q) \vee (p \wedge r)$

XI. $p \vee (q \wedge r)$ eq. $(p \vee q) \wedge (p \vee r)$

XII. $(p \rightarrow q) \wedge (p \rightarrow r)$ eq. $p \rightarrow (q \wedge r)$

XIII. $(p \rightarrow q) \vee (p \rightarrow r)$ eq. $p \rightarrow (q \vee r)$

XIV. $(p \rightarrow r) \wedge (q \rightarrow r)$ eq. $(p \vee q) \rightarrow r$

XV. $(p \rightarrow r) \vee (q \rightarrow r)$ eq. $(p \wedge q) \rightarrow r$

XVI. $(p \wedge q) \rightarrow r$ eq. $p \rightarrow (q \rightarrow r)$

XVII. $(p \rightarrow q) \rightarrow \left[(p \wedge r) \rightarrow (q \wedge r)\right]$ eq.

$(p \rightarrow q) \rightarrow \left[(p \vee r) \rightarrow (q \vee r)\right]$

The statements I through VII are especially important. Let P be any formula with $p_1, p_2, \ldots, p_n$ as prime components. From the way the term formula was defined, the only connectives which appear in it are $\neg$,

$\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$. By applying statement VII above, P can be written in an equivalent form say P_1 using only the connectives $\neg$, $\vee$, $\wedge$, and $\rightarrow$. Then applying statement I above, P_1 can be written in an equivalent form say P_2 using only $\neg$, $\vee$, and $\wedge$. In the application of statement I, it was necessary to negate certain prime components in order to obtain the equivalent form. Finally, by applying statement IV, P_2 can be written in equivalent form, say P_3, using only $\neg$ and $\vee$ and again negating appropriate prime components. Thus any formula P can be written in equivalent form using only the connectives $\neg$ and $\vee$.

Some certain occurrences of the prime components in the formula P may have to be negated in order to obtain the equivalent form, while others may not. Thus the equivalent form may have more prime components than P has. In fact, it may have at most 2n prime components. The prime components may be written $p_1, p_2, \ldots, p_n, p_{n+1}, \ldots, p_k$ where p_{n+i} is $\neg p_i$ as may required by the procedure above.

An interesting property which some classical tautologies have in the three-valued logic will now be presented. For emphasis, the property will be stated in the form of a theorem.

<u>Theorem 1.1</u>. Suppose P is any formula with prime components $p_1, p_2, \ldots, p_n$ and can be written by the above procedure as $p_1 \vee p_2 \vee \ldots \vee p_n \vee p_{n+1} \vee \ldots \vee p_k$. Then if P is a classical tautology, P has no F values in the three-valued logic.

Proof: Suppose P is a formula with prime components $p_1, p_2, \ldots, p_n$ which can be written equivalently as $p_1 \vee p_2 \vee \ldots \vee p_n \vee p_{n+1} \vee \ldots \vee p_k$. Assume that P is a classical tautology. Suppose for some assignment α to $p_1, p_2, \ldots p_n$ in the three-valued logic, P has the value F. Now α must assign to at least one of the prime components the value M, or else all would have value T or F, and for this assignment, P would have value T since it is a classical tautology. Now since one of the prime components say p_i has value M, then even if all the other prime components had value F, $p_1 \vee p_2 \vee \ldots \vee p_n \vee p_{n+1} \ldots \vee p_k$ would still have value M. This is clearly a contradiction to the assumption that P had value F for the assignment α. Thus any classical tautology which can be written in the form above will have no F values in the three-valued logic.

It is believed that a more general form of Theorem 1.1 is true, that is, that no classical tautology has F values in the three-valued logic. However, a proof has not been established for this more general statement.

It is interesting to observe the relationship between formulas P and Q being equivalent and the formula $P \longleftrightarrow Q$ being a tautology. In the classical logic, P eq. Q if and only if $\models P \longleftrightarrow Q$. This, however, is not the case in the three-valued logic. A weaker relationship holds in the three-valued logic as the next theorem will state. But first an observation must be made. Any formula P with $p_1, p_2, \ldots p_n$ as prime components will have value M whenever all the prime components have value M. This property is apparent from observing the basic truth table. Note that the last row in the table assigns to each of $\neg p$, $p \wedge q$, $p \vee q$, $p \rightarrow q$, and $p \leftrightarrow q$ the value M. Hence any combination of these connectives will also have value M. The following lemma must be proved before the theorem can be stated.

Lemma 1.1. If P and Q are any formulas such that $\models P \leftrightarrow Q$, then P and Q have the same prime components.

Proof: Suppose P and Q are any formulas with prime components $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_m$ respectively, and $\models P \leftrightarrow Q$. Suppose Q has a prime component say q_i which does not appear in $p_1, p_2, \ldots, p_n$. Then for any assignment of values which assigns M to each of $p_1, p_2, \ldots, p_n$ and a value T or F to q_i, P has value M. Now, regardless of what value Q has for this assignment, $P \leftrightarrow Q$ does not have value T (refer to basic truth table for $\leftrightarrow$). This contradicts the assumption that $\models P \leftrightarrow Q$. Thus each prime component of Q must appear in $p_1, p_2, \ldots, p_n$. A similar argument shows that each prime component of P must appear in $q_1, q_2, \ldots, q_m$.

Lemma 1.1 does not hold in the classical logic. For example, $\models p \leftrightarrow [p \wedge (q \vee \neg q)]$ in the classical sense, but the prime component q does not appear in the prime formula p. The theorem can now be stated and proved.

Theorem 1.2. If P and Q are any formulas such that $\models P \leftrightarrow Q$, then P eq. Q.

Proof: Suppose P and Q are any formulas such that $\models P \leftrightarrow Q$. By lemma 1.1, P and Q have the same prime components say $p_1, p_2, \ldots, p_n$. Now for any assignment to $p_1, p_2, \ldots, p_n$ which does not assign to each p_i the value M, $P \leftrightarrow Q$ has value T. Since interest is concentrated on the connective $\leftrightarrow$, P and Q may be treated as prime formulas. From the basic truth table observe that $P \leftrightarrow Q$ has value T only when P and Q are both T or both F. So for any assignment to $p_1, p_2, \ldots, p_n$ for which they do not all have value M, P and Q have the same truth value. Hence for those assignments, their truth tables are identical. It has already been observed that whenever $p_1, p_2, \ldots, p_n$ all have value M, P and Q both have value M. Thus the truth tables for P and Q are identical. So P eq. Q.

Notice that the converse of theorem 1.2 does not hold in the three-valued logic. It has been shown that $(p \vee q) \vee r$ eq. $p \vee (q \vee r)$. However, when p and q have value F and r has value M, then $(p \vee q) \vee r$ and $p \vee (q \vee r)$ both have value M which means that $[(p \vee q) \vee r] \leftrightarrow [p \vee (q \vee r)]$ has value M and hence is not a tautology.

Another interesting relationship between tautologies and equivalent statements is given in the following theorem.

<u>Theorem 1.3</u>. If the formula P is such that $\models P$ and P eq. Q where P and Q have the same prime components, then $\models Q$.

Proof: Suppose P and Q are any formulas with the same prime components say $p_1, p_2, \ldots, p_n$ such that $\models$P and P eq. Q. For any assignment to $p_1, p_2, \ldots, p_n$ which does not assign to all $p_1, p_2, \ldots, p_n$ the value M, P has the value T. Since P eq. Q, Q also has value T for this assignment to $p_1, p_2, \ldots, p_n$.

This concludes the introduction to this three-valued logic. In the next chapter, the notion of consequence will be defined for the three-valued logic, and its properties will be compared with those of consequence in the classical two-valued logic. Also the behavior of formulas formed by substitution of formulas for prime components in a given formula will be investigated.

CHAPTER II

SUBSTITUTION AND CONSEQUENCE

The properties of substitution which hold in the classical logic will now be examined in the setting of the three-valued logic. Some of the corresponding properties remain true in the three-valued logic, and some do not.

One of the properties of substituting formulas for prime components in a formula is the following. Suppose P is any formula with prime components $p_1, p_2, \ldots, p_n$, and let P* be the result of substituting formula Q for each occurrence of the prime component p_i. Now, in the classical logic it is the case that if $\models P$, then $\models P^*$. This, however, is not the case in the three-valued logic. Let P be the formula $(p \vee \neg p) \vee (q \vee \neg q)$. The reader can easily verify that P is a tautology. By substituting $p \wedge q$ for each occurrence of p in P, P* is then the formula $[(p \wedge q) \vee \neg(p \wedge q)] \vee (q \vee \neg q)$. But whenever p has value T and q has value M, P* has value M and hence is not a tautology. Hence, substitution in the three-valued logic does not preserve tautologies as it does in the classical logic.

Another property of substitution in the classical logic states that if T_P is a formula with a specific occurrence of formula P and if T_Q is the result of substituting Q for P in T_P and P eq. Q, then T_P eq. T_Q. This property holds in the three-valued logic and is stated in the following theorem.

<u>Theorem 2.1</u>. If T_P is any formula with a specific occurrence of formula P, and if T_Q is the result of substituting formula Q for P in T_P, and P eq. Q, then T_P eq. T_Q.

Proof: Suppose T_P is a formula with a specific occurrence of formula P. Suppose formula Q is such that P eq. Q and T_Q is the result of substituting Q for P in T_P. Let $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_m$ be the prime components of P and Q respectively. Since P eq. Q, P and Q have the same truth value for any given assignment of values to $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m$.

Let $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m, t_1, t_2, \ldots, t_j$ be the prime components of T_P considered as a function of n + m + j variables. For any assignment of values to $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m, t_1, t_2, \ldots, t_j$. T_P has a given value as a function of $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m, t_1, t_2, \ldots, t_j$. For this assignment, P as a function of n + m variables has a specific truth value. Since P eq. Q, for this assignment to $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m$, Q will have the same value as P. Now if Q is substituted for P in T_P, the result is T_Q. And since P and Q have the same truth value for any given assignment to $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m$, then T_Q whose prime components are $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m, t_1, t_2, \ldots, t_j$ will have the same value as T_P for any assignment of values to $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m, t_1, t_2, \ldots, t_j$. Hence T_P eq. T_Q.

Employing the notation of Theorem 2.1, another property of substitution which holds in the classical logic is the following. If $\vDash P \leftrightarrow Q$, then $\vDash T_P \leftrightarrow T_Q$. This property does not hold in the three-valued logic. An example for which the above property does not hold is the following. It has been shown that $\vDash (p \wedge q) \rightarrow (p \vee q)$ and $\vDash (p \rightarrow q) \vee (q \rightarrow p)$, thus both formulas have value T whenever p and q do not both have value M. Thus $\vDash \left[(p \wedge q) \rightarrow (p \vee q)\right] \leftrightarrow \left[(p \rightarrow q) \vee (q \rightarrow p)\right]$. However, $\left[(p \wedge q) \rightarrow (p \vee q)\right] \wedge r \leftrightarrow \left[(p \rightarrow q) \vee (q \rightarrow p)\right] \wedge r$ is not a tautology since the formula has value M whenever p and q have value M and r has value T.

A property which closely related to the one discussed here does hold in the three-valued logic. It is stated in the next theorem again using the notation of Theorem 2.1.

<u>Theorem 2.2</u>. If $\vDash P \leftrightarrow Q$ and $\vDash T_P$, then $\vDash T_Q$.

Proof: Suppose P and Q are any formulas such that $\vDash P \leftrightarrow Q$. Also, suppose that T_P is a formula such that $\vDash T_P$. Now since $\vDash P \leftrightarrow Q$, by Lemma 1.1, P and Q have the same prime components. Therefore T_P and T_Q must have the same prime components. By Theorem 1.2, since $\vDash P \leftrightarrow Q$, then P eq. Q. Thus by Theorem 2.1, T_P eq. T_Q. Now, by Theorem 1.3, $\vDash T_Q$.

Note that under the hypothesis of Theorem 2.2, it is the case that $\vDash T_P \leftrightarrow T_Q$ since both T_P and T_Q have value T whenever not all the prime components of either have value M.

Now as a prelude to the discussion of the notion of consequence, a theorem from classical logic which allows one to generate classical tautologies will be stated. The theorem states that if P and Q are any formulas such that $\models P$ and $\models P \rightarrow Q$, then $\models Q$. Although the theorem may not be as useful in the three-valued logic in terms of generating tautologies, it is nevertheless true. It is stated here primarily as motivation for the definition of consequence.

Theorem 2.3. If P and Q are any formulas such that $\models P$ and $\models P \rightarrow Q$, then $\models Q$.

Proof: The proof is by contradiction. Suppose P and Q are any formulas such that $\models P$ and $\models P \rightarrow Q$. Let $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_m$ be their prime components respectively. Let $t_1, t_2, \ldots, t_k$ be the totality of the prime components of P and Q. The contradiction will be presented now in two cases.

Case 1. Suppose that for some assignment of truth values to $q_1, q_2, \ldots, q_m$ such that not all have value M, Q has value M. This assignment of values can be extended to also assign values to $p_1, p_2, \ldots, p_n$. If $p_1, p_2, \ldots, p_n$ all have value M, then P has value M. But then $P \rightarrow Q$ has value M. This is a contradiction to the assumption that $\models P \rightarrow Q$. If $p_1, p_2, \ldots, p_n$ do not all have value M, then P has value T. But then $P \rightarrow Q$ has value M. This again contradicts the assumption $\models P \rightarrow Q$. Thus whenever $q_1, q_2, \ldots, q_m$ do not all have value M, Q does not have value M.

Case 2. Suppose that for some assignment of truth values to $q_1, q_2, \ldots, q_m$ such that not all have value M, Q has value F. Again, this assignment can be extended to assign values to all of $p_1, p_2, \ldots, p_n$. Now, if $p_1, p_2, \ldots, p_n$ all have value M, then P has value M. But since Q has value F, $P \rightarrow Q$ has value M. This contradicts the assumption $\models P \rightarrow Q$. If $p_1, p_2, \ldots, p_n$ do not all have value M, then P has value T. But since Q has value F, then $P \rightarrow Q$ has value F. This contradicts once again the assumption $\models P \rightarrow Q$.

Thus whenever $q_1, q_2, \ldots, q_m$ are not all assigned the value M, Q must have value T. Hence, $\models Q$.

Note that in the notation of Theorem 2.3, whenever P is true and $P \rightarrow Q$ is true, then Q is true. Thus it is said that Q is a consequence of P and $P \rightarrow Q$. In other words since P, and $P \rightarrow Q$ are tautologies, Q must also be a tautology.

The general definition of consequence is stated as follows. The formula Q is said to be a consequence of formulas $P_1, P_2, \ldots, P_n$ if for any assignment to the prime components of $P_1, P_2, \ldots, P_n$, and Q which makes $P_1, P_2, \ldots, P_n$ have value T, then for this assignment, Q also has value T. This definition is the same as the definition of consequence in the classical logic. The notation used to indicate that Q is a consequence of $P_1, P_2, \ldots, P_n$ is $P_1, P_2, \ldots, P_n \models Q$. In the classical logic, there are certain conditions which guarantee that Q is a consequence of $P_1, P_2, \ldots, P_n$. The problem is to determine whether or not similar conditions will guarantee consequence in the three-valued logic.

One of these conditions in the classical logic which guarantees consequence is the following statement. If P and Q are any formulas then $\models P \rightarrow Q$ if and only if $P \models Q$. This relationship holds partially in the three-valued logic.

<u>Theorem 2.4</u>. If P and Q are any formulas such that $\models P \rightarrow Q$, then $P \models Q$.

Proof: Suppose P and Q are any formulas such that $\models P \rightarrow Q$. Let $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_m$ be the prime components of P and Q respectively and let $t_1, t_2, \ldots, t_k$ be the totality of prime components which appear in P and Q. Suppose α is any assignment of truth values to $t_1, t_2, \ldots, t_k$ for which the formula P has value T. Since P has value T for the assignment α, α does not assign the value M to all of $t_1, t_2, \ldots, t_k$. Hence $P \rightarrow Q$ has value T for this assignment α. Thus for the assignment α, Q must have value T or else $P \rightarrow Q$ would not have value T. Hence, $P \models Q$.

Note that the converse of this theorem does not hold in the three-valued logic. The reader can easily verify that $(p \vee q) \models (p \vee q) \vee r$. But when p has value M, q has value F, and r has value M, then $(p \vee q) \rightarrow [(p \vee q) \vee r]$ has value M and is therefore not a tautology.

A condition in classical logic which guarantees that a formula Q is a consequence of a set of formulas say

$P_1, P_2, \ldots, P_n$ is the following. $P_1, P_2, \ldots, P_n \models Q$ if and only if $P_1 \wedge P_2 \wedge \ldots \wedge P_n \models Q$. This condition holds in the three-valued logic and is stated in the next theorem.

<u>Theorem 2.5</u>. If $P_1, P_2, \ldots, P_n$, and Q are any formulas, then $P_1, P_2, \ldots, P_n \models Q$ if and only if $P_1 \wedge P_2 \wedge \ldots \wedge P_n \models Q$.

Proof: Suppose $P_1, P_2, \ldots, P_n$ and Q are any formulas such that $P_1, P_2, \ldots, P_n \quad Q$. Suppose $t_1, t_2, \ldots, t_k$ are the prime components of $P_1, P_2, \ldots, P_n \quad Q$. Let α be any assignment of truth values to $t_1, t_2, \ldots, t_k$ for which $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ has value T. Now for this assignment, $P_1, P_2, \ldots, P_n$ must each have value T since $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ has value T. Since $P_1, P_2, \ldots, P_n$ all have value T, and $P_1, P_2, \ldots, P_n \models Q$, then Q must also have value T. Thus for any assignment for which $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ has value T, Q has value T. Hence, $P_1 \wedge P_2 \wedge \ldots \wedge P_n \models Q$.

Now suppose $P_1 \wedge P_2 \wedge \ldots \wedge P_n \models Q$. Let β be any assignment of truth values to $t_1, t_2, \ldots, t_k$ for which $P_1, P_2, \ldots, P_n$ all have value T. Then for any such assignment β, since $P_1, P_2, \ldots, P_n$ all have value T, $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ has value T. And since $P_1 \wedge P_2 \wedge \ldots \wedge P_n \models Q$, for any such assignment β, Q has value T. Thus for any assignment for which $P_1, P_2, \ldots, P_n$ all have value T, Q also has value T. Hence, $P_1, P_2, \ldots, P_n \models Q$.

Note that in Theorem 2.5, $P_1 \; P_2 \; . \; . \; . \; P_n$ is actually one formula whose prime components are precisely the prime components of $P_1, P_2, \ldots, P_n$. In view of this observation, a trivial corollary is obtained.

<u>Corollary 2.5.1</u>. Adopting the notation of Theorem 2.5, if $\models \left[P_1 \wedge P_2 \wedge \ldots \wedge P_n\right] \rightarrow Q$, then $P_1, P_2, \ldots, P_n \models Q$.

Proof: Suppose $P_1, P_2, \ldots, P_n$ and Q are formulas such that $\models \left[P_1 \wedge P_2 \wedge \ldots \wedge P_n\right] \rightarrow Q$. Then by Theorem 2.4, $P_1 \wedge P_2 \wedge \ldots \wedge P_n \models Q$. Now by Theorem 2.5, $P_1, P_2, \ldots, P_n \models Q$.

Another condition in the classical logic which guarantees that a formula Q is a consequence of $P_1, P_2, \ldots, P_n$ states that $P_1, P_2, \ldots, P_n \models Q$ if and only if $P_1, P_2, \ldots, P_{n-1} \models P_n \rightarrow Q$. This statement is only partially true in the three-valued logic as the next theorem indicates.

<u>Theorem 2.6</u>. If $P_1, P_2, \ldots, P_n$ and Q are any formulas, and if $P_1, P_2, \ldots, P_{n-1} \models P_n \rightarrow Q$, then $P_1, P_2, \ldots, P_n \models Q$.

Proof: Suppose $P_1, P_2, \ldots, P_n$ and Q are any formulas such that $P_1, P_2, \ldots, P_{n-1} \models P_n \rightarrow Q$. Suppose $t_1, t_2, \ldots, t_k$ are the prime components of $P_1, P_2, \ldots, P_n$ and Q. Let α be any assignment of truth values to $t_1, t_2, \ldots, t_k$ for which $P_1, P_2, \ldots, P_n$ all have value T. Thus for this assignment α, $P_n \rightarrow Q$ must have value T since $P_1, P_2, \ldots, P_{n-1} \models P_n \rightarrow Q$. And since

P_n has value T, Q must also have value T for this assignment α. Thus for any assignment to $t_1, t_2, \ldots, t_k$ for which $P_1, P_2, \ldots, P_n$ all have value T, Q must also have value T. Hence, $P_1, P_2, \ldots, P_n \models Q$.

Note that the converse of Theorem 2.6 fails to hold in the three-valued logic. The reader can verify that $(p \rightarrow q), (q \rightarrow r) \models (p \rightarrow r)$. However, when p and q have value T, and r has value M, then $p \rightarrow q$ has value T, but $(q \rightarrow r) \rightarrow (p \rightarrow r)$ has value M. Thus it is not the case that $(p \rightarrow q) \models (q \rightarrow r) \rightarrow (p \rightarrow r)$.

In the classical logic, Theorem 2.4 is generalized to the following statement. $P_1, P_2, \ldots, P_n \models Q$ if and only if $\models P_1 \rightarrow (P_2 \rightarrow (\ldots (P_n \rightarrow Q) \ldots))$. A partial generalization of Theorem 2.4 in the three-valued logic can be made.

Theorem 2.7. If $P_1, P_2, \ldots, P_n$ and Q are any formulas such that $\models P_1 \rightarrow (P_2 \rightarrow (\ldots (P_n \rightarrow Q) \ldots))$, then $P_1, P_2, \ldots, P_n \models Q$.

Proof: The proof is by induction on the number of formulas labled P_i. Suppose $P_1, P_2, \ldots, P_n$ and Q are any formulas such that $\models P_1 \rightarrow (P_2 \rightarrow (\ldots (P_n \rightarrow Q) \ldots))$. Now for n = 1, the statement above becomes $\models P_1 \rightarrow Q$. And by Theorem 2.4, then $P_1 \models Q$.

Suppose the theorem is satisfied for n = k. Now for n = k + 1, the hypothesis of the theorem is $\models P_1 \rightarrow (P_2 \rightarrow (\ldots (P_k \rightarrow (P_{k+1} \rightarrow Q)) \ldots))$. Since the theorem is true for n = k, then $P_1, P_2, \ldots, P_k \models (P_{k+1} \rightarrow Q)$. But by

Theorem 2.6 it must be true that $P_1, P_2, \ldots, P_k, P_{k+1} \models Q$. Thus for any natural n, if $\models P_1 \rightarrow (P_2 \rightarrow (\ldots (P_n \rightarrow Q) \ldots))$, then $P_1, P_2, \ldots, P_n \models Q$.

The converse of Theorem 2.7 does not hold in the three-valued logic. As before, $(p \rightarrow q), (q \rightarrow r) \models p \rightarrow r$, but when p has value T, q has value M, and r has value F, the formula $(p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$ has value M, and hence is not a tautology.

In classical logic, the rules of inference for the statement calculus are based on two important properties. These properties also hold in the three-valued logic, and their proof is similar to the proof in the classical logic.

<u>Theorem 2.8</u>. If $P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots, Q_k$, and R are any formulas, then

I. $P_1, P_2, \ldots, P_n \models P_i$ for $i = 1, 2, \ldots, n$.

II. If $P_1, P_2, \ldots, P_n \models Q_j$ for $j = 1, 2, \ldots, k$, and if $Q_1, Q_2, \ldots, Q_k \models R$, then $P_1, P_2, \ldots, P_n \models R$.

Proof: Suppose $P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots, Q_k$, and R are any formulas. Then I above is clearly an immediate result of the definition of consequence.

For II above, suppose $t_1, t_2, \ldots, t_h$ are the prime components which appear in one of $P_1, P_2, \ldots, P_n$, $Q_1, Q_2, \ldots, Q_k$, and R. And suppose that $P_1, P_2, \ldots, P_n \models Q_j$ for $j = 1, 2, \ldots, k$, and $Q_1, Q_2, \ldots, Q_k \models R$. Let α be any assignment of truth values to $t_1, t_2, \ldots, t_h$ for which each of $P_1, P_2, \ldots, P_n$ have value T.

Since $P_1, P_2, \ldots, P_n \models Q_j$ for $j = 1, 2, \ldots, k$, for the assignment α, each of $Q_1, Q_2, \ldots, Q_k$ must have value T. But then since $Q_1, Q_2, \ldots, Q_k \models R$, for this assignment α to $t_1, t_2, \ldots, t_h$, R must have value T. Thus for any assignment of truth values to $t_1, t_2, \ldots, t_h$ for which $P_1, P_2, \ldots, P_n$ all have value T, R has value T. Hence, $P_1, P_2, \ldots, P_n \models R$.

To summarize the theorems and observations discussed in this chapter, it can be said that most of the properties of substitution in the classical logic also hold in the three-valued logic when restricted or modified in some way. Similarly for the properties of consequence. However, there is a central difference between the conditions of consequence for the classical logic and the corresponding conditions in the three-valued logic. In the classical logic, most of the conditions are biconditional statements. In the three-valued logic, with the exception of Theorem 2.5, the statements are only conditional rather than biconditional as they are in the classical logic.

In the following chapter, truth functions of two arguments will be investigated. The definitions of the slash and dagger functions from classical logic will be extended for the three-valued logic. Also a truth function will be defined which will generate all the truth functions of two arguments in the three-valued logic.

CHAPTER III

GENERATING FUNCTIONS

A topic of interest in the classical logic is that of generating functions. That is, functions of two arguments which when composed with themselves in various ways will generate all possible truth functions of two arguments. In classical logic there are two such functions. The behavior of these two generating functions will be investigated in the three-valued logic. Also a function will be defined which will generate all the possible truth functions of two arguments in the three-valued logic.

In the classical logic, there are 16 different truth functions of two variables. The two truth functions which generate all of the 16 truth functions are the slash ($/$) function and the dagger ($\downarrow$) function. The definition of these functions can be extended for the three-valued logic as the following truth table illustrates.

p	q	p/q	p $\downarrow$ q
T	T	F	F
T	F	T	F
F	T	T	F
F	F	T	T
T	M	M	F
F	M	T	M
M	T	M	F
M	F	T	M
M	M	M	M

When the definitions of the slash and dagger functions are extended as above, then both the slash and the dagger functions generate each of the usual functions $\neg p$, $p \vee q$, $p \wedge q$, and $p \longrightarrow q$ in the three-valued logic. The next two theorems are the statements of this property. The proofs are by examining truth tables. Some will be left to the reader.

Theorem 3.1. If p / q is defined as in the truth table above, then

I. $p \wedge q$ eq. $(p / q) / (p / q)$

II. $\neg p$ eq. p / p

III. $p \vee q$ eq. $(p / p) / (q / q)$

IV. $p \rightarrow q$ eq. $[(p / p) / (p / p)] / [q / q]$

Proof of I: Examine the truth table.

p	q	$p \wedge q$	p / q	$(p / q) / (p / q)$
T	T	T	F	T
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F
T	M	M	M	M
F	M	F	T	F
M	T	M	M	M
M	F	F	T	F
M	M	M	M	M

Hence $p \wedge q$ eq. $(p / q) / (p / q)$

Proof of II: Again examine the truth table.

p	p	p p
T	F	F
F	T	T
M	M	M

Proofs of III and IV can be obtained in a similar way and are left to the reader.

Thus any formula in the three-valued logic can be written equivalently using only the slash function. This will also be true of the dagger function, but first a theorem for the dagger function similar to Theorem 3.1 must be stated.

Theorem 3.2. If $p \downarrow q$ is defined as in the truth table above, then

I. $p \vee q$ eq. $(p \downarrow q) \downarrow (p \downarrow q)$

II. $\neg p$ eq. $p \downarrow p$

III. $p \wedge q$ eq. $(p \downarrow p) \downarrow (q \downarrow q)$

IV. $p \rightarrow q$ eq. $[(p \downarrow p) \downarrow q] \downarrow [(p \downarrow p) \downarrow q]$

Proof of I: Examine the truth table.

p	q	$p \vee q$	$p \downarrow q$	$(p \downarrow q) \downarrow (p \downarrow q)$
T	T	T	F	T
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F
T	M	T	F	T
F	M	M	M	M
M	T	T	F	T
M	F	M	M	M
M	M	M	M	M

Hence $p \vee q$ eq. $(p \downarrow q) \downarrow (p \downarrow q)$

Proof of II: Again examine the truth table.

p	$\neg p$	$p \downarrow p$
T	F	F
F	T	T
M	M	M

Proofs of III and IV are obtained in a similar way and are left to the reader.

As in the case of the slash function, any formula in the three-valued logic can be written equivalently using only the dagger function. However neither the slash function, the dagger function, nor any combination of the two can generate all the truth functions of two arguments in the three-valued logic. For example, neither nor both together can generate the function of two arguments which maps the pair (T,T) into the element M. For when p and q both have value T, any combination of the slash or dagger function or both, will have value T or F.

In the three-valued logic, there are 19,683 different truth functions of two arguments. The problem now is to find one of these functions which when composed with itself in various ways will generate each of the other truth functions. There exist such a function. It will be called the square function of two arguments written $p \square q$. The only motivation for the definition of the function was to define it in such a way that $(p \square p) \square p$ eq. $\neg p$. The function will now be defined by means of a truth table, and some composition functions will be developed to familiarize the reader with the behavior of the function under composition.

p	q	$p \square q$	$p \square p$	$q \square q$	$(q \square q) \square (p \square p)$	$(p \square p) \square p$
T	T	M	M	M	F	F
T	F	F	M	M	F	F
F	T	T	M	M	F	T
F	F	M	M	M	F	T
T	M	T	M	F	M	F
F	M	M	M	F	M	T
M	T	F	F	M	T	M
M	F	T	F	M	T	M
M	M	F	F	F	M	M

p	q	$[p \square p] \square [(q \square q) \square (p \square p)]$	$(p \square p) \square (q \square q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	T	F
T	M	F	T
F	M	F	T
M	T	T	M
M	F	T	M
M	M	M	M

p	q	$\{(p \square p) \square (q \square q)\} \square \{(p \square p) \square [(q \square q) \square (p \square p)]\}$
T	T	T
T	F	T
F	T	T
F	F	T
T	M	F
F	M	F
M	T	F
M	F	F
M	M	F

p	q	$\{(p \square p) \square [(q \square q) \square (p \square p)]\} \square \{(p \square p) \square (q \square q)\}$
T	T	F
T	F	F
F	T	F
F	F	F
T	M	T
F	M	T
M	T	T
M	F	T
M	M	F

To ease notational pains, some notation for the composition functions must be introduced. Certain functions which will be used repeatedly will be designated by upper case letters. Also, if some function X is "squared" with itself, the result will be written R(X) to indicate $X \square X$. If the negation of some function X is desired, it can be obtained by the function $(X \square X) \square X$. Thus the notation

$\neg X$ will represent the function $(X \square X) \square X$. Since only one function is being composed, the symbol $\square$ will often be left out of the notation, i.e., $p \square q$ will be written pq. Two functions will now be designated by A and B, and others will be designated as the need arises. Let $A = \{(p \square p) \square (q \square q)\} \square \{(p \square p) \square [(q \square q) \square (p \square p)]\}$ and let $B = \{(p \square p) \square [(q \square q) \square (p \square p)]\} \square \{(p \square p) \square (q \square q)\}$. Their truth tables are calculated above.

Another composition function which will be used repeatedly is developed in the following truth table.

p	q	$p \square q$	$p \square (p \square q)$	$(p \square q) \square p$	$[p \square (p \square q)] \square [(p \square q) \square p]$
T	T	M	F	F	M
T	F	F	F	T	T
F	T	T	T	F	F
F	F	M	M	T	F
T	M	T	M	M	F
F	M	M	M	T	F
M	T	F	T	M	T
M	F	T	F	T	T
M	M	F	T	M	T

Now let $A_1 = [p \square (p \square q)] \quad [(p \square q) \square p]$.

Some special functions which will be very important in the later development will now be developed. They are the "constant functions". That is, those functions which map each ordered pair of $W \times W$ where $W = \{T, M, F\}$ into the same element of W. Their truth table is given below.

p	q	A	$R(A)$	$R[R(A)]$	$R(A) \square R[R(A)]$
T	T	T	M	F	T
T	F	T	M	F	T
F	T	T	M	F	T
F	F	T	M	F	T

p	q	A	R(A)	R[R(A)]	R(A) $\square$ R[R(A)]
T	M	F	M	F	T
F	M	F	M	F	T
M	T	F	M	F	T
M	F	F	M	F	T
M	M	F	M	F	T

These functions will also be referred to as the M-function, the F-function, and the T-function respectively.

Now another special group of equally important functions will be generated. These are functions which are constant except for one ordered pair of the domain. One such function is the function which maps all the ordered pairs into the element M except the ordered pair (M, M) which is mapped onto F. Some functions whose functional values vary for only one ordered pair will be developed and then a procedure will be described by which all functions whose values differ for only one ordered pair can be developed. The truth tables for some of these functions are given below.

p	q	A	B	A $\square$ B	R(A $\square$ B)
T	T	T	F	F	M
T	F	T	F	F	M
F	T	T	F	F	M
F	F	T	F	F	M
T	M	F	T	T	M
F	M	F	T	T	M
M	T	F	T	T	M
M	F	F	T	T	M
M	M	F	F	M	F

Let A_0 = R(A $\square$ B)

p	q	A_0	$\neg A_0$	pq	qp	$(pq)(qp)$	$R\left[(pq)(qp)\right]$	$\neg A_0 \square R\left[(pq)(qp)\right]$
T	T	M	M	M	M	F	M	F
T	F	M	M	F	T	T	M	F
F	T	M	M	T	F	F	M	F
F	F	M	M	M	M	F	M	F
T	M	M	M	T	F	F	M	F
F	M	M	M	M	T	F	M	F
M	T	M	M	F	F	M	F	T
M	F	M	M	T	M	T	M	F
M	M	F	T	F	F	M	F	F

Let $B_0 = \neg A_0 \square R\left[(pq)(qp)\right]$.

The complete truth table for these functions becomes increasingly lengthy and for that reason will be omitted from this paper. The reader may convince himself of their validity by merely calculating the truth tables.

p	q	$\left[A_0 \square R(A_0)\right] \square R\left[(qp)q\right]$
T	T	T
T	F	T
F	T	T
F	F	T
T	M	F
F	M	T
M	T	T
M	F	T
M	M	T

Let $C_0 = \left[A_0 \square R(A_0)\right] \square R\left[(qp)q\right]$.

p	q	$\left\{\left[\left\{A_0 \square R(A_0)\right\} \square R(A_0)\right] \square C_0\right\} \square R\left\{\left[q(pq)\right]\ \left[(qp)q\right]\right\}$
T	T	T
T	F	T
F	T	T
F	F	T
T	M	T
F	M	T
M	T	T
M	F	F
M	M	T

Let $D_0 = \Big\{\Big[\big[\{A_0 \square R(A_0)\} \square R(A_0)\big] \quad C_0\Big\} \square R\Big\{[q(pq)] \quad [(qp)q]\Big\}$.

p	q	$\{R(A_0) \square [A_0 \square R(A_0)]\} \square \{R[(qq)(pp)] \square \neg B_0\}$
T	T	M
T	F	M
F	T	M
F	F	M
T	M	M
F	M	T
M	T	M
M	F	M
M	M	M

Let $E_0 = \{R(A_0) \square [A_0 \quad R(A_0)]\} \square \{R[(qq)(pp)] \square \neg B_0\}$.

p	q	$\{(([A_0 \square R(A_0)] \square R(A_0)) \square D_0\} \square R[q(qp)]$
T	T	T
T	F	T
F	T	T
F	F	F
T	M	T
F	M	T
M	T	T
M	F	T
M	M	T

Let $F_0 = \{(([A_0 \square R(A_0)] \square R(A_0)) \square D_0\} \square R[q(qp)]$.

p	q	$\neg B_0 \square \{[A_0 \square R(A_0)] \square R[(pq)(qq)]\}$
T	T	M
T	F	F
F	T	M
F	F	M
T	M	M
F	M	M
M	T	M
M	F	M
M	M	M

Let $G_0 = \neg B_0 \square \{[A_0 \square R(A_0)] \square R[(pq)(qq)]\}$.

p	q	$\{G_0 \square R[(A_0 \square R(A_0)) \square R(A_0)]\} \square (\{((\{A_0 \square R(A_0)\} \square R(A_0)) \square F_0\} \square R[p(qp)])$
T	T	T
T	F	T
F	T	M
F	F	T
T	M	T
F	M	T
M	T	T
M	F	T
M	M	T

Let $H_0 = \{G_0 \square R[(A_0 \square R(A_0)) \square R(A_0)]\} \square (\{((\{A_0 \square R(A_0)\} \square R(A_0)) \square F_0\} \square R[p(qp)])$.

p	q	$R(A_1)$
T	T	F
T	F	M
F	T	M
F	F	M
T	M	M
F	M	M
M	T	M
M	F	M
M	M	M

Let $I_0 = R(A_1)$.

It is now a matter of calculation to show that all of the truth functions similar to A_0, B_0, . . . , I_0 can be generated by "squaring" the given functions on the left or right by some combination of the constant functions. The truth tables for those functions similar to I_0 are given below. The others can be obtained in a similar manner and are left to the reader.

p	q	$R(A_1)$	$\neg R(A_1)$	$R(A)$	$R(A) \blacksquare R(A_1)$	$R(A_1) \blacksquare R(A)$
T	T	F	T	M	T	M
T	F	M	M	M	F	F
F	T	M	M	M	F	F
F	F	M	M	M	F	F
T	M	M	M	M	F	F
F	M	M	M	M	F	F
M	T	M	M	M	F	F
M	F	M	M	M	F	F
M	M	M	M	M	F	F

p	q	$R(R(A))$	$R(A_1) \blacksquare R(R(A))$	$R(R(A)) \blacksquare R(A_1)$
T	T	F	F	M
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T
T	M	F	T	T
F	M	F	T	T
M	T	F	T	T
M	F	F	T	T
M	M	F	T	T

Thus all the possible truth functions which have the same functional value on all the ordered pairs except on (T, T) are generated by the square function. By a similar procedure, the other functions similar to the one above may be generated.

Another important collection of functions which are generated by the square function is the collection of functions which have a two element range. That is, those functions which map each ordered pair into either of two elements. There is a general scheme by which any such function can be obtained. However, first a procedure will be developed by which any function whose range consists of the elements T and F can be obtained.

Suppose X is any function of two arguments whose range consists of the elements T and F. And, suppose that $\{(u_i, v_i) : i = 1, 2, \ldots, n, n \leq 8\}$ is the set of ordered pairs for which X has value F. Now, "square" the T-function on the left by the function which has value F for (u_1, v_1) and has value F for the other ordered pair, and calculate the negation of the result. This procedure generates a function X_1 which has value T for 8 ordered pairs, and has value F for (u_1, v_1). Now square X_1 on the left by the function which has value F for (u_2, v_2) and value M for the other ordered pairs, and calculate the negation of the result. This procedure generates a function X_2 which has value F for (u_1, v_1) and (u_2, v_2) and has value T for all other ordered pairs. Repeating this procedure at most n times the function X is finally obtained.

As before, to obtain the other functions similar to functions whose range consists of the elements T and F, the function can be squared on the left or right by appropriate constant functions. So to generate any function whose range consists of two elements, first generate an appropriate function whose range consists of the elements T and F. Now square this function on the left or right by the appropriate constant functions to obtain the desired function.

Now there is a simple two step procedure which will generate any function whose range consists of three elements. Suppose Y is any such function, and suppose $\{(u_i, v_i)\}$ is the set of ordered pairs for which Y has value F and

$\{(\bar{u}_j, \bar{v}_j)\}$ is the set of ordered pairs for which Y has value M. First square the T-function on the left by the function which has value M for $\{(u_i, v_i)\}$ and has value F for all other ordered pairs. Now square the resulting function on the left by the function which has value T on $\{(\bar{u}_j, \bar{v}_j)\}$ and has value M for all other ordered pairs. Now calculate the negation of this result to obtain a function equivalent to Y.

Thus the square function generates all the truth functions of two arguments in the three-valued logic.

CHAPTER IV

SUMMARY

The three-valued logic developed in this paper has many properties of the classical logic. With the notion of tautology as defined in this paper, several classical tautologies which would not be tautologies from the classical viewpoint, are readily obtainable in the three-valued logic. Also the concepts of equivalence, consequence, substitution and tautology are interrelated in some of the same ways as they are in the classical logic. However, many of the theorems which are biconditional in the classical logic are only conditional in the three-valued logic. Also several of the properties of substitution do not hold in the three-valued logic. Nevertheless, many of the properties of classical logic are preserved in this three-valued logic.

There are two main problems that are motivated by this paper. One problem which is only partially solved in this paper may warrant further study. That is proving or disproving the statement that no classical tautology, regardless of what form it is in, has F values in the three-valued logic. Theorem 3.1 states that classical tautologies which can be written in a certain form have no F values in the three-valued logic, but the more general statement has not been proven. Another problem which is not discussed in this paper but which may be worthy of consideration, is that of axiomatizing the three-valued logic. Consideration may also be given to the feasibility of developing some rules of inference and a predicate calculus. In reality, this paper probably motivates more questions than it answers.

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