ON THE OPEN CONVEXITY OF NEURAL CODES WITH FIVE NEURONS

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by

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DEDICATION

For my Mother and Father.

ABSTRACT

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The process through which the brain interprets and understands environmental spatial information has been a major area of interest in the field of Neuroscience. The brain, through the use of receptive fields and corresponding neurons called "place cells", naturally segments its environment into distinct regions of space. This process of segmenting and labeling regions can be abstracted into the concept of a Neural Code, which is a special set of labels called "codewords" that describe these distinct regions. Given the layout of a set of receptive fields, it is easy to generate the corresponding Neural Code. However finding a corresponding set of open convex receptive fields given a Neural Code remains an open problem. Recent research in this area has lead to the classification of all Neural Codes which have an open convex realization for codes of up to four Neurons. In this thesis, we begin a classification of codes with an open convex realization for five Neurons.

KEY WORDS: Neural Code, Convex, Neuron, Classification, Open

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CHAPTER 1

Introduction

Our brains accomplish the complex of task and navigating the physical world every day, but how exactly is this achieved? Research into this topic lead neuroscientists John O'Keefe and Johnathan Dostrovsky to the discovery of place cells within the hippocampus of the brain in 1971 [8]. All neurons have a corresponding receptive field, that is, a particular region of the stimulus space (e.g. the surface of the skin, the visual field) in which stimulus, such as touching, will alter the firing rate of that neuron. The altered firing rate gives the brain information which it can then use to react to its environment. A place cell is a special type of neuron whose receptive field is a physical region in space (often called the "place field"). When an animal enters a place field (for example, the corner of a room), the corresponding place cell within the brain will fire at a higher rate than when the animal is not in that place field. It is believed that the brain uses this information to build a spatial mapping of its surrounding environment, but how this information is used to build such a map, and what information this neural activity can actually tell the brain remains an open area of research.

Of course the brain does not only have access to the information given by place cells. It is more likely that the brain uses this information in conjunction with other environmental stimulus to accurately determine its location within the spatial map. However the question still remains as to how much information place cell activity alone can actually provide. In particular, because a place field is a particular region of space, it is clear that place fields must somehow segment an animals environment into distinct regions identifiable purely through neural activity. One may describe these divisions using a neural code. A neural code is a special set of labels which describes these regions using information about the firing patterns of a corresponding set of place cells and their receptive fields. Although a neural code may be determined if place cell activity is already known, we are more interested in the reverse question; given a neural code, what can we say about the topological structure of place fields described by the code? In particular, we are interested in whether or not the place fields of a given code are allowed to be open convex regions of Euclidean space. This is because repeated experiments have observed place fields to be approximately open convex regions of space (as seen in Figure 1). It is this question of which neural codes are *convex codes* that this thesis is focused on.

Previous research has already partially answered this question. Curto *et. al.*[2, 3], and Giusti and Itskov [5] used combinatorial topology and commutative algebra to provide a complete classification of convex codes on up to four neurons. Giusti and Itskov introduced the concept of a "local obstruction" to convexity, a trait characteristic of non-convex codes. They were able to show that codes which had a local obstruction were necessarily non-convex. Curto *et. al.* achieved the complete classification of convex codes on up to four neurons by first organizing codes according to their associated simplicial complex, and then showing that for each simplicial complex Δ, there was a set of "mandatory codewords" whose presence in a neural code (with corresponding simplicial complex Δ) is required to avoid a local obstruction. Thus if a code is convex, it must contain the mandatory codewords of its simplicial complex. For codes on up to four neurons, it was shown that the convex codes are exactly the codes with no local obstructions.

Recently it was shown by Lienkaemper, Shiu, and Woodstock [7] that having no local obstructions is not a strong enough criterion to guarantee convexity of a neural code. To show this, they provided an explicit code on five neurons which did not exhibit any local

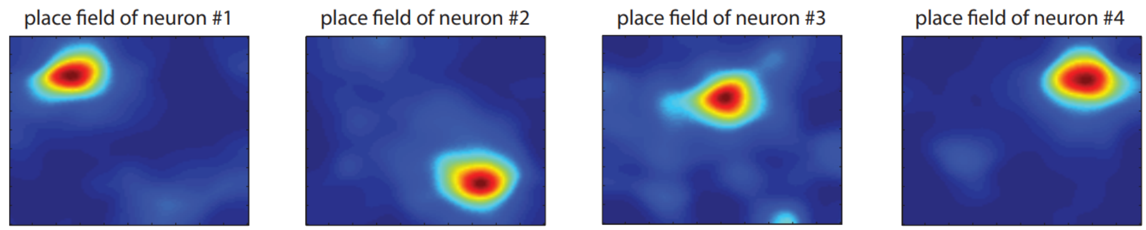


Figure 1: Four place fields for four CA1 pyramidal neurons (place cells) inside the hippocampus of a rat. Data was recorded while the rat explored a 1.5m $\times$ 1.5m box. Blue areas indicate low to no activity from the neuron, while red areas indicate high activity from the neuron. These images were computed from data provided by the Pastalkova lab, which is described in [6].

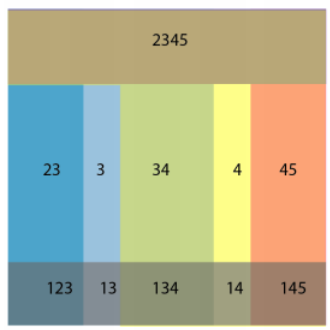


Figure 2: A drawing of the receptive fields and codewords for the code which was shown to have no local obstructions and also be non-convex in [7]. Note the receptive field for neuron 4 (which is the union of all regions with 4 in their label) is a non-convex set.

obstructions to convexity, and was also not realizable with open convex receptive fields. (A non-convex realization of the code's receptive fields can be seen in Figure 2.) Thus the question of which codes are convex remains open for cases involving more than four neurons. Our goal is to classify all open convex codes on five neurons. Given that local obstructions are not strong enough to determine convexity, it is not clear what exactly the underlying factors that determine if a code is convex are. Our hope is that completing such a classification will provide insight into what other criteria may be at play in determining a codes convexity.

In Chapter 2, we will discuss preliminary definitions and important theorems related to the study of open convex neural codes, and neural codes in general. Chapter 3 will contain the main results of this thesis, namely an examination of which codes on five neurons are open convex.

CHAPTER 2

Definitions and Background

In this chapter we will introduce the essential definitions, assumptions, and notation. Within the context of a neural code, we will interpret neuron activity as binary; a neuron is either active or inactive. We define $[n]$ to be the set of integers from 1 to n, i.e. $[n] := \{1,2,\ldots,n\}$.

2.1 Neural Codes

Definition 1. A **neural code** $\mathscr{C}$ on n neurons is a set of subsets of $[n]$ such that $\emptyset \in \mathscr{C}$. Elements of $\mathscr{C}$ are called *codewords*. A codeword of $\mathscr{C}$ is said to be *maximal* if it is not a proper subset of any other codeword in $\mathscr{C}$.

Biologically, a codeword may be thought of as a label for a specific region of space. More specifically, it is the region of space in which only those neurons indexed by the codeword are active. The empty set is always a codeword by way of convention, as it simplifies our assumptions. Alternatively one may generate a neural code directly from a set of receptive fields in the following way:

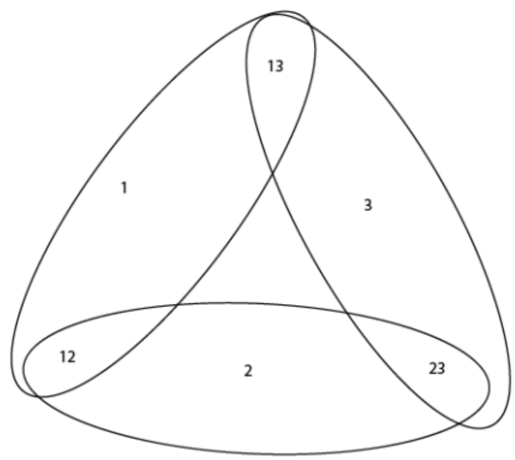
Let X be a topological space and $\mathscr{U} = \{U_1,\ldots,U_n\}$ be a collection of subsets of X with the property that $\bigcup_{i=1}^{n} U_i \subsetneq X$. For $\sigma \subseteq [n]$, we define $U_\sigma := \bigcap_{i\in\sigma} U_i$. Then $\mathscr{U}$ has associated with it the code $\mathscr{C}(\mathscr{U})$, defined as

$$\mathscr{C}(\mathscr{U}) := \left\{\sigma \subseteq [n] : U_\sigma \setminus \bigcup_{j\in[n]\setminus\sigma} U_j \neq \emptyset\right\}.$$

In this characterization, the topological space X represents the *stimulus space* of the neurons, while the collection $\mathscr{U}$ represents the set of receptive fields which correspond to a set of n place neurons in the brain. It is important that the union of the sets in $\mathscr{U}$ be properly contained in X, as this guarantees that $\emptyset \in \mathscr{C}(\mathscr{U})$.

It is not hard to generate a neural code from a given collection $\mathscr{U}$ as this amounts to simply checking a condition for each subset of $[n]$, so a natural question is how difficult is the reverse? That is, given a neural code $\mathscr{C}$, can one always find a space X and a collection $\mathscr{U}$ which *realizes* this code? This motivates the following definition.

Definition 2. Let $\mathscr{C}$ be a neural code, let X be a topological space, and let $\mathscr{U} = \{U_1, \dots, U_n\}$ be a collection of subsets of X. We say $\mathscr{U}$ **realizes** $\mathscr{C}$ if $\mathscr{C} = \mathscr{C}(\mathscr{U})$.



Example 3. [7] A realization of the code $\mathscr{C} = \{\{1,2\},\{1,3\},\{2,3\},\{1\},\{2\},\{3\},\emptyset\}$. Here our stimulus space is $\mathbb{R}^2$, so our receptive fields are open subsets of the plane. The distinct regions cut out by this realization have been labeled with their corresponding codewords. The receptive fields are the three oval sets, U_1 (top left), U_2 (bottom), and U_3 (top right).

We now show that without any restrictions on the sets in our collection $\mathscr{U}$, being a realizable code bears no significance.

Lemma 4. *[1] Every code $\mathscr{C}$ may be realized by a collection $\mathscr{U}$ of subsets of $\mathbb{R}$.*

Proof. Let $\mathscr{C}$ be a neural code on n neurons. For each nonempty codeword $\sigma \in \mathscr{C}$, choose a point $x_\sigma \in \mathbb{R}$ such that for every pair $\sigma, \tau \in \mathscr{C}$, $x_\sigma = x_\tau$ if and only if $\sigma = \tau$. For each $i \in [n]$, define $U_i = \{x_\sigma | i \in \sigma\}$. It is straightforward to check that the collection $\mathscr{U} = \{U_1, \ldots, \mathscr{U}_n\}$ realizes $\mathscr{C}$. □

It is clear from this result that we must place restrictions on the collection $\mathscr{U}$ if we wish to learn anything of note. One thing to notice is that receptive fields which look like the sets constructed in the previous lemma (finite collections of distinct points) are unlikely to exist. Experimental observations from multiple sources have shown receptive fields of place neurons to be approximately convex regions of space. This motivates our first restriction.

2.2 Convexity

By convention, we will from now on assume our stimulus space to be $\mathbb{R}^d$, equipped with the standard Euclidean topology.

Definition 5. A set $E \subseteq \mathbb{R}^d$ is **convex** if for any pair of points $a, b \in E$, we have $c \in E$ if there exists $\lambda \in [0,1]$ such that $c = (1-\lambda)a + \lambda b$.

Intuitively a convex set in $\mathbb{R}^d$ is one which contains any line segment that begins and ends within the set. We now define *convex codes*.

Definition 6. Let $\mathscr{C}$ be a neural code. We say $\mathscr{C}$ is a **convex code** if there exists a collection, $\mathscr{U}$, of subsets of $\mathbb{R}^d$ such that $\mathscr{U}$ realizes $\mathscr{C}$ and every set in $\mathscr{U}$ is convex.

A natural question now is which codes are convex? This question was recently answered by the following result.

Theorem 7. *[4] Every code $\mathscr{C}$ may be realized by a collection $\mathscr{U}$ of convex subsets of $\mathbb{R}^d$, for some d.*

Thus being a convex code by the previous definition bears no distinction, and so again we must further restrict our collection $\mathscr{U}$. While there are many possible restrictions to make, our goal is to have the sets in our collection $\mathscr{U}$ resemble as close as possible the actual structure of a receptive field. We now define the type of code we are concerned with classifying.

Definition 8. Let $\mathscr{C}$ be a neural code. We say $\mathscr{C}$ is an **open convex code** if there exists a collection, $\mathscr{U}$, of subsets of $\mathbb{R}^d$ such that $\mathscr{U}$ realizes $\mathscr{C}$ and every set in $\mathscr{U}$ is open and convex. The smallest d for which an open convex realization exists is called the *minimal embedding dimension*.

The problem of determining if a code is open convex is one of interest, as so far we have only been able to find sufficient and necessary conditions for open convexity for cases that limit the number of neurons in the code. Open convexity is also the first restriction so far that is non-trivial. This is illustrated by the following result.

Lemma 9. *The code $\mathscr{C} = \{\{1,2\},\{1,3\},\emptyset\}$ on three neurons is not open convex.*

Proof. Suppose $\mathscr{U} = \{U_1, U_2, U_3\}$ is a collection of open sets which realizes $\mathscr{C}$. We will show U_1 cannot be convex. First note that $U_1 \subseteq U_2 \cup U_3$, as otherwise $\{1\}$ would be a codeword of $\mathscr{C}$. Second note that $U_2 \cup U_3 \subseteq U_1$, as otherwise $\{2\}, \{3\}$, or $\{2,3\}$ would be codewords of $\mathscr{C}$. Thus $U_1 = U_2 \cup U_3$. Finally note that $U_2 \cap U_3 = \emptyset$, as otherwise $\{1,2,3\}$ would be a codeword of $\mathscr{C}$. Now suppose $a, b \in U_1$ such that $a \in U_2$ and $b \in U_3$. Because U_2 and U_3 are disjoint open sets, a line connecting points a and b cannot be contained in $U_2 \cup U_3$. Since $U_2 \cup U_3 = U_1$, it follows that U_1 cannot be convex. $\square$

For the rest of this paper we will use an abbreviated notation when explicitly writing a neural code. For example, in the above lemma the code $\mathscr{C}$ would be written as $\{\mathbf{12}, \mathbf{13}, \emptyset\}$, with maximal codewords in bold. Another notation used in exterior works referenced by this paper is that of binary strings (i.e. $\mathscr{C} = \{110, 101, 000\}$). For consistency, we will forego this notation.

It should be noted that the intersection patterns of convex sets is already a well-established subject. The key difference in our study is the emphasis on all the particular regions cut out by the convex sets as well as which sets intersect.

2.3 Simplicial Complexes

Before we discuss our methods for determining which codes on five neurons are open convex, we will first flesh out previously established results on the topic. We begin with the definition of a simplicial complex.

Definition 10. An (abstract) **simplicial complex** Δ on n vertices is a nonempty collection of subsets of $[n]$ that is closed under inclusion. More precisely, if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. Elements of Δ are called *faces*. The faces of Δ which are not contained in any other faces of Δ are called *facets*.

Topologically, a simplex is a generalization of the triangle to higher dimensions and a simplicial complex is a structure which consists of various simplexes which intersect at certain faces. There are two simplicial complexes we are mainly concerned with: the simplicial complex of a neural code, and the *link* of a face of a simplicial complex.

Definition 11. Let $\mathscr{C}$ be a neural code. We define the simplicial complex associated with this code as

$$\Delta(\mathscr{C}) := \{\sigma \subseteq [n] : \sigma \subseteq c \text{ for some } c \in \mathscr{C}\}.$$

If $\mathscr{C}$ is a code realized by some collection $\mathscr{U}$, then $\Delta(\mathscr{C})$ carries with it topological information regarding the sets in the collection $\mathscr{U}$. In fact, one may show that $\Delta(\mathscr{C})$ is equivalent to the nerve of $\mathscr{U}$, from which many previously established topological results apply.

Definition 12. Let Δ be a simplicial complex, and σ a face of Δ. Then the **link** of σ with respect to Δ is defined as

$$\mathrm{Lk}_{\Delta}(\sigma) := \{\omega \in \Delta : \omega \cup \sigma \in \Delta \text{ and } \omega \cap \sigma = \emptyset\}$$

Intuitively the link of a face σ within a given simplicial complex Δ is the set of faces in Δ which are "linked" to σ by another face. Both of these simplicial complexes play an important part in the discussion of local obstructions.

2.4 Local Obstructions

A local obstruction is a topological condition which prevents a code from being open convex. Previous results have shown that this condition is independent from the choice of realization of a code, and in fact may be completely characterized by a code's simplicial complex.

Definition 13. [2] Let $\mathscr{C}$ be a neural code. Let $\mathscr{M} = \{M_1, \ldots, M_m\}$ be the set of facets of $\Delta(\mathscr{C})$. We say $\mathscr{C}$ has a **local obstruction** if there exists a face σ of $\Delta(\mathscr{C})$ with the following properties:

1. $\sigma = \bigcap_{i \in I} M_i \neq \emptyset$ for some $I \subseteq [n]$.

2. $\sigma \notin \mathscr{C}$.

3. $\mathrm{Lk}_{\Delta(\mathscr{C})}(\sigma)$ is not contractible.

Here the link of σ with respect to Δ is not contractible if its geometric realization is not contractible. The most important result in relation to local obstructions is as follows:

Theorem 14. *[5] Let $\mathscr{C}$ be a neural code. If $\mathscr{C}$ has a local obstruction, then $\mathscr{C}$ is **not** open convex.*

Thus at a bare minimum a neural code must at least have no local obstructions in order to possibly be convex. In fact it was shown for $n \leq 4$ that having no local obstructions was also sufficient to guarantee convexity:

Theorem 15. *[7] $\mathscr{C}$ be a neural code on $n \leq 4$ neurons. Then the following are equivalent:*

1. *$\mathscr{C}$ is open convex.*

2. *$\mathscr{C}$ is **max-intersection-complete**: If $\sigma \subseteq [n]$ is the intersection of maximal codewords of $\mathscr{C}$, then $\sigma \in \mathscr{C}$.*

3. *$\mathscr{C}$ has no local obstructions.*

For $n > 4$, we only have that $2 \Rightarrow 1 \Rightarrow 3$.

In fact, both the implications $3 \Rightarrow 1$ and $3 \Rightarrow 2$ fail in the $n = 5$ case. Specifically, it was shown that the code $\mathscr{C}4 = \{\mathbf{2345}, \mathbf{123}, \mathbf{134}, \mathbf{145}, 13, 14, 23, 34, 45, 3, 4, \emptyset\}$ has no local obstructions and is also not open convex and not max-intersection-complete [7].

We observe from our definition of a local obstruction that a code may avoid such obstructions to convexity by simply including any facet intersection with a non-contractible link. Since these codewords depend only on the simplicial complex of the code, this gives rise to the following characterization of a code with no local obstructions:

Theorem 16. *[7] Let $\mathscr{C}$ be a neural code. Then there exists a code, $\mathscr{C}_{\min}(\Delta(\mathscr{C}))$ called the minimal code of $\Delta(\mathscr{C})$, which depends only on the simplicial complex of $\mathscr{C}$ and has the following property: $\mathscr{C}$ has no local obstructions if and only if $\mathscr{C}_{\min}(\Delta(\mathscr{C})) \subseteq \mathscr{C}$.*

This characterization provides a new way to check for the absence of local obstructions in a code: for a code on n neurons, find the minimal code for each simplicial complex on n vertices. Then check if the code contains the minimal code of its associated simplicial complex. Note it is not necessary that the minimal code of some simplicial complex contain all facet intersections. Consider the previously mentioned code $\mathscr{C}4$. It is in fact the minimal code of its associated simplicial complex but does not contain the codeword $1 = \mathbf{123} \cap \mathbf{134} \cap \mathbf{145}$.

The most valuable result we shall use in the classification of the $n = 5$ case is the following:

Theorem 17. *[1] Let $\mathscr{C}$ be an open convex neural code. Then for any code $\mathscr{C}_0$ for which $\mathscr{C} \subseteq \mathscr{C}_0 \subseteq \Delta(\mathscr{C})$, $\mathscr{C}_0$ is open convex.*

This result lets us classify entire sections of neural codes under a specific simplicial complex as being open convex. Out strategy then is to the find the smallest neural code within a given simplicial complex for which there exists an open convex realization. Since a code must at least contain the minimal code of its simplicial complex, we will mainly focus on finding open convex realizations for minimal codes. In cases where the minimal code is not open convex, our goal is to find the minimum set of additional codewords that must be added to the minimal code for an open convex realization to exist. Note by considering the contrapositive statement of this theorem, we may also classify entire sections of codes under a given simplicial complex as being non-open-convex by finding the *largest* code that is non-open-convex, as any subset of such a code with the same simplicial complex must necessarily be non-open-convex as well.

CHAPTER 3

Main Results

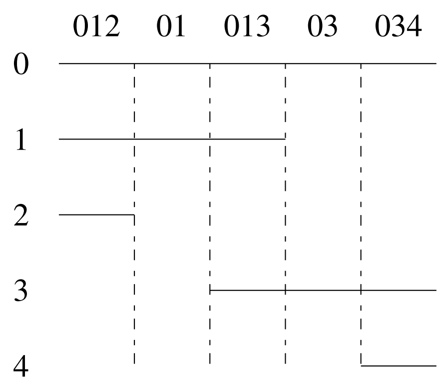
Previous considerations of the $n = 5$ case succeeded in cataloging the minimal codes of the 157 possible simplicial complexes on 5 vertices [7]. The minimal codes found can be divided into two categories: those which are max-intersection-complete and those which are not. From Theorem 15, we know that max-intersection-complete codes are convex. Thus the only minimal codes for which open convexity is currently unknown are those which are not max-intersection-complete. Of the 157 simplicial complexes on 5 vertices, 22 contain such a minimal code:

Table 1: The 22 minimal codes on fives neurons which are not max-intersection-complete, classified by their maximal and mandatory codewords [7]. A mandatory codeword is a facet intersection with a noncontractible link. The full code is the union of the maximal and mandatory codewords, as well as the empty set.

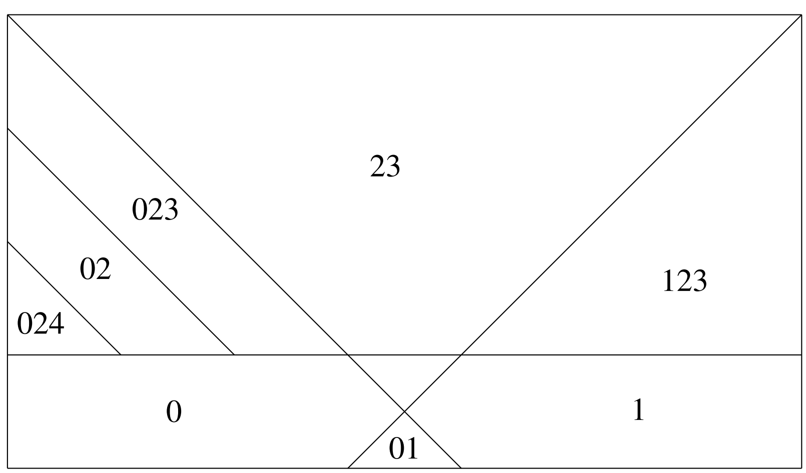
Label	Maximal c.w.	Mandatory c.w.
$\mathscr{C}1$	{012, 013, 034}	{03, 01}
$\mathscr{C}2$	{023, 123, 024, 01}	{02, 23, 0, 1}
$\mathscr{C}3$	{023, 124, 234, 01}	{24, 23, 0, 1}
$\mathscr{C}4$	{1234, 012, 034, 013}	{12, 34, 13, 03, 01, 1, 3}
$\mathscr{C}5$	{012, 123, 034, 013}	{13, 12, 03, 01, 1}
$\mathscr{C}6$	{014, 123, 034, 012}	{12, 04, 01, 3}
$\mathscr{C}7$	{134, 034, 123, 02, 01}	{34, 13, 0, 1, 2}
$\mathscr{C}8$	{034, 024, 124, 23, 01}	{24, 04, 0, 1, 2, 3}
$\mathscr{C}9$	{023, 034, 123, 024, 01}	{04, 23, 02, 03, 0, 1}
$\mathscr{C}10$	{023, 134, 123, 024, 01}	{23, 02, 13, 0, 1, 4}
$\mathscr{C}11$	{023, 134, 234, 024, 01}	{24, 34, 23, 02, 0, 1, 2}
$\mathscr{C}12$	{023, 013, 014, 012, 234}	{23, 02, 03, 01, 0, 4}
$\mathscr{C}13$	{034, 013, 012, 123, 024}	{12, 04, 02, 13, 03, 01, 0, 1}
$\mathscr{C}14$	{034, 013, 014, 012, 123}	{12, 04, 13, 03, 01, 0, 1}
$\mathscr{C}15$	{034, 014, 012, 123, 234}	{12, 04, 34, 23, 01}
$\mathscr{C}16$	{034, 234, 124, 02, 13, 01}	{24, 34, 0, 1, 2, 3}
$\mathscr{C}17$	{134, 034, 123, 124, 01, 02}	{12, 14, 34, 13, 0, 1, 2}
$\mathscr{C}18$	{023, 034, 123, 024, 124, 01}	{24, 12, 04, 23, 02, 03, 0, 1, 2}
$\mathscr{C}19$	{023, 134, 123, 234, 024, 01}	{24, 34, 23, 02, 13, 0, 1, 2, 3}
$\mathscr{C}20$	{034, 013, 014, 012, 123, 024}	{12, 04, 02, 13, 03, 01, 0, 1}
$\mathscr{C}21$	{034, 013, 014, 012, 123, 234}	{12, 04, 34, 23, 13, 03, 01, 0, 1, 3}
$\mathscr{C}22$	{034, 013, 024, 124, 014, 012, 123}	{24, 12, 04, 14, 02, 13, 03, 01, 0, 1, 2, 4}

Thus we focus on the open convexity of these 22 codes. We note that for minimal codes with an open convex realization, having no local obstructions is sufficient to guarantee open convexity. We now present our results for each of these minimal codes.

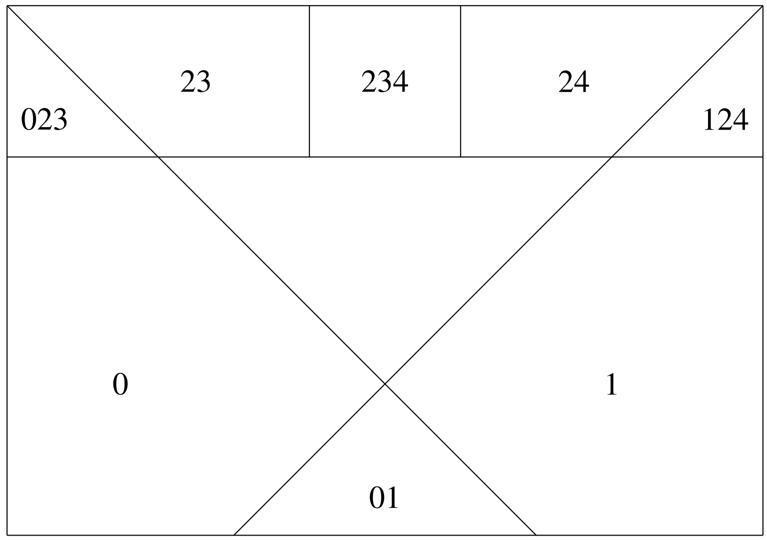
A convex realization of $\mathscr{C}1$ in $\mathbb{R}^1$. Here the lines represent open intervals.



A convex realization of $\mathscr{C}2$ in $\mathbb{R}^2$.

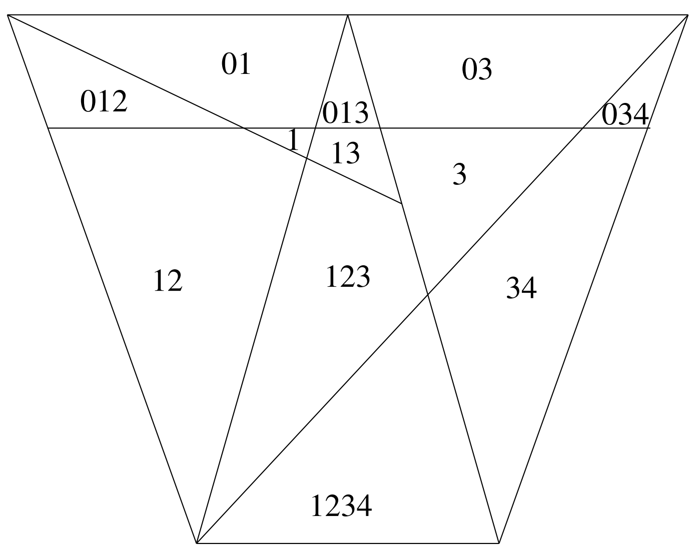


A convex realization of $\mathscr{C}3$ in $\mathbb{R}^2$.

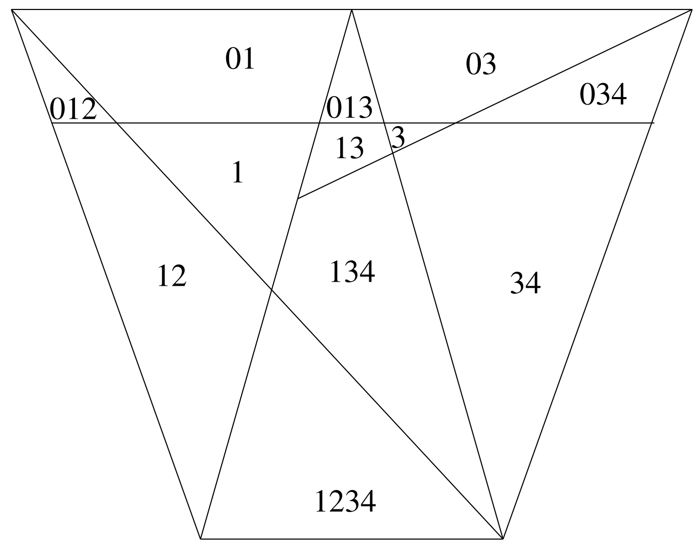


Code $\mathscr{C}4$ was shown to be non-convex by Lienkaemper, Shiu, and Woodstock [7]. As mentioned in their paper, the only codewords which "break" their method for proving $\mathscr{C}4$ non-convex are 123 and 134. While they provide a convex realization of $\mathscr{C}4 \cup \{123, 134\}$, we provide convex realizations for both $\mathscr{C}4 \cup \{123\}$ and $\mathscr{C}4 \cup \{134\}$.

A convex realization of $\mathscr{C}4 \cup \{123\}$ in $\mathbb{R}^2$:

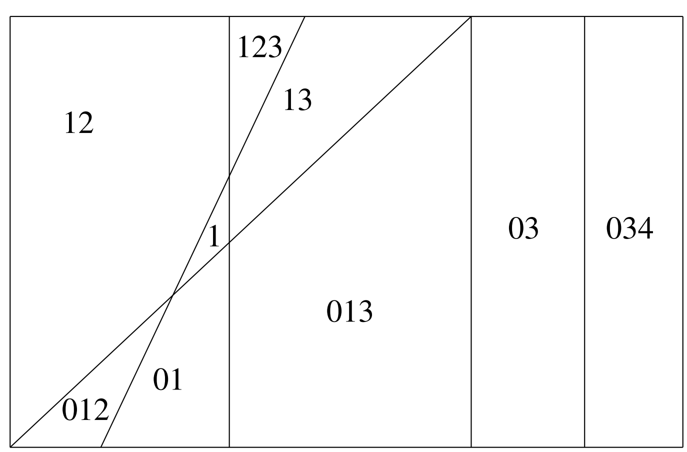


A convex realization of $\mathscr{C}4 \cup \{134\}$ in $\mathbb{R}^2$:

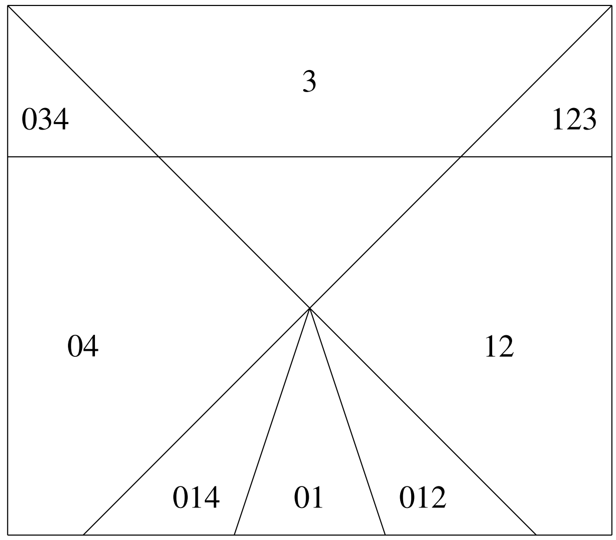


Noting that $\Delta(\mathscr{C}4) \setminus \{123, 134\}$ is non-convex via the proof given in [7] for the non-convexity of $\mathscr{C}4$, by Theorem 17 we have that any subset code is necessarily non-convex as well. This completes the classification for all codes within $\Delta(\mathscr{C}4)$.

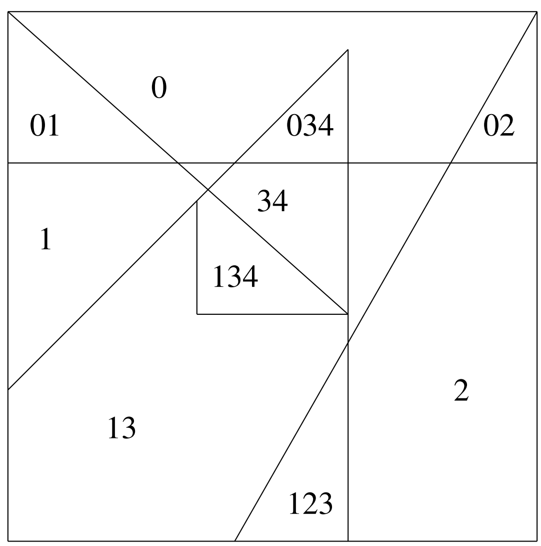
A convex realization of $\mathscr{C}5$ in $\mathbb{R}^2$.



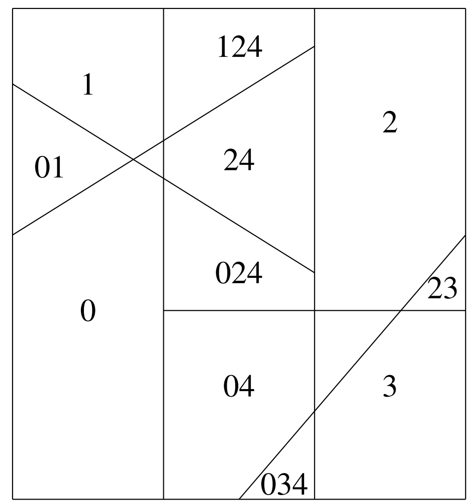
A convex realization of $\mathscr{C}6$ in $\mathbb{R}^2$.



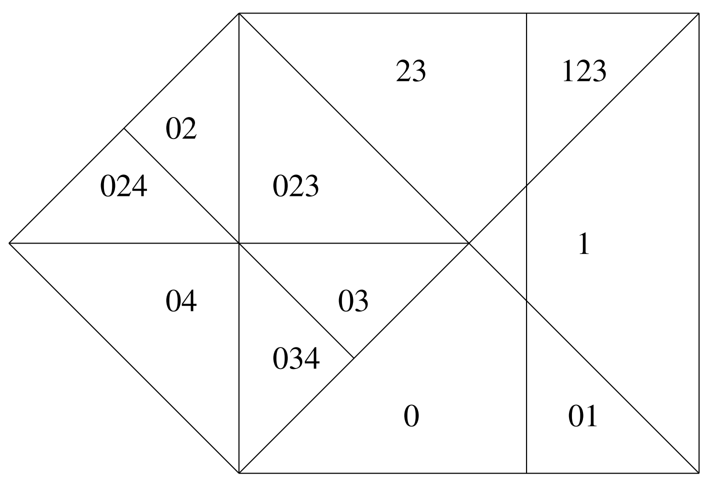
A convex realization of $\mathscr{C}7$ in $\mathbb{R}^2$.



A convex realization of $\mathscr{C}8$ in $\mathbb{R}^2$.

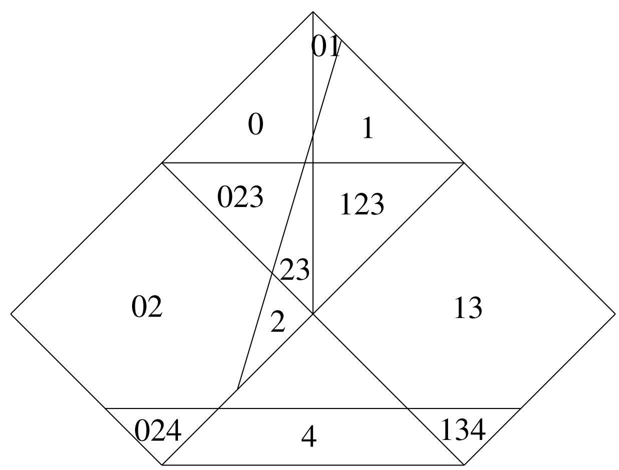


A convex realization of $\mathscr{C}9$ in $\mathbb{R}^2$.

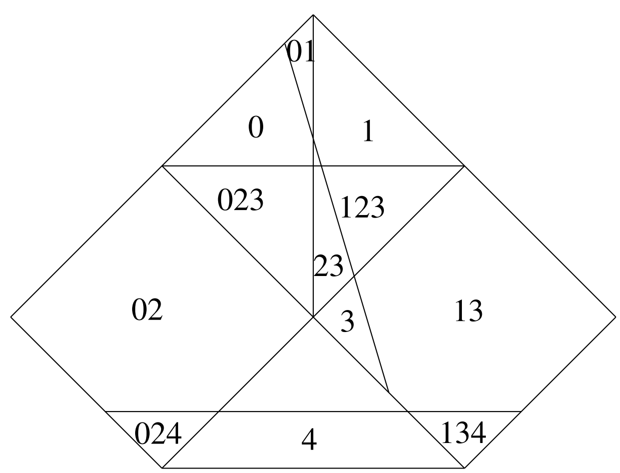


Code $\mathscr{C}10$ is the only code for which we were unable to construct a convex realization in $\mathbb{R}^2$ or $\mathbb{R}^3$. We conjecture that this code is in fact non-convex, and provide convex realizations for this code when codewords 2 or 3 are added. (Note adding both would make the code max-intersection-complete, and thus open convex by theorem 15.)

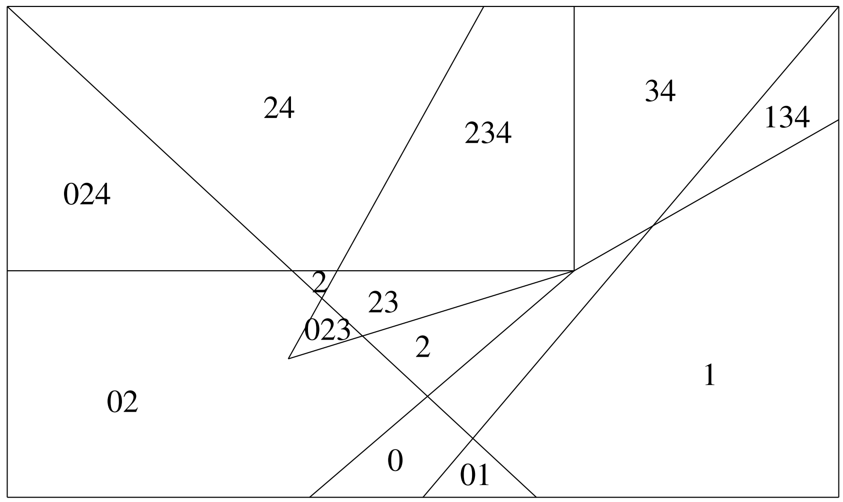
A convex realization of $\mathscr{C}10 \cup \{2\}$ in $\mathbb{R}^2$.



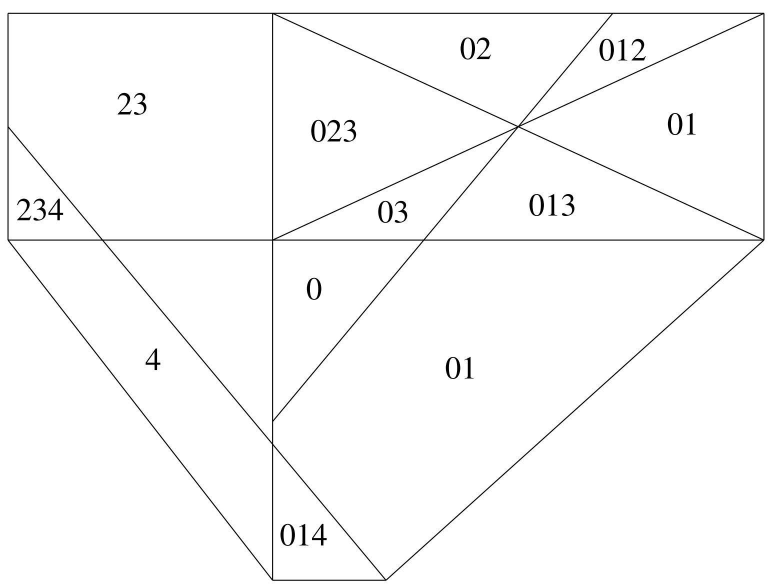
A convex realization of $\mathscr{C}10 \cup \{3\}$ in $\mathbb{R}^2$.



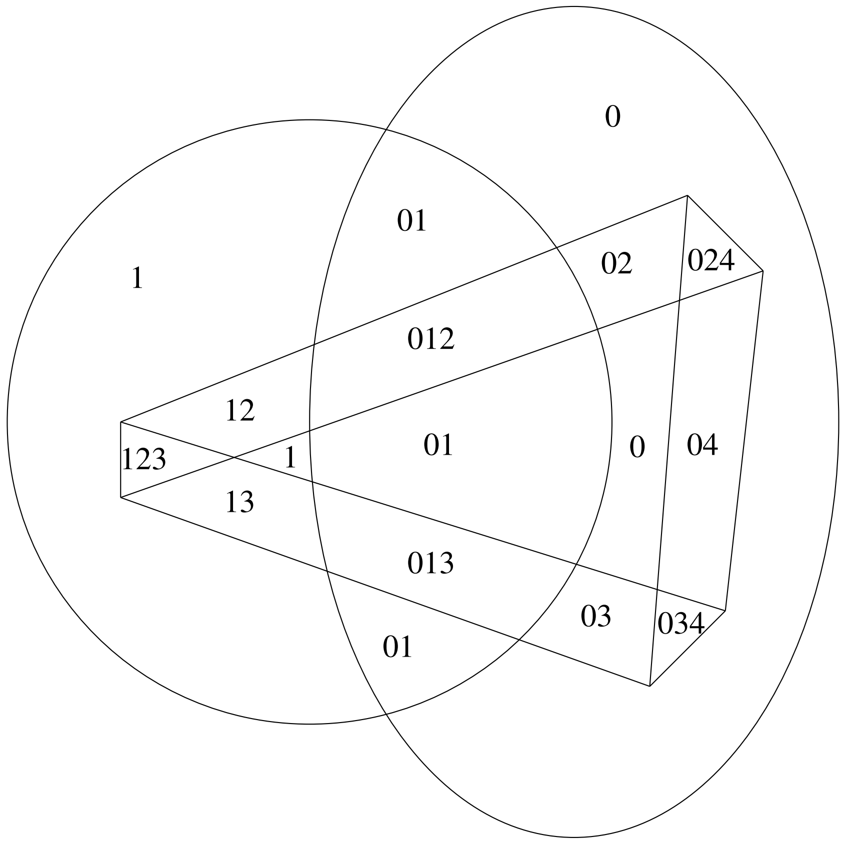
A convex realization of $\mathscr{C}11$ in $\mathbb{R}^2$.



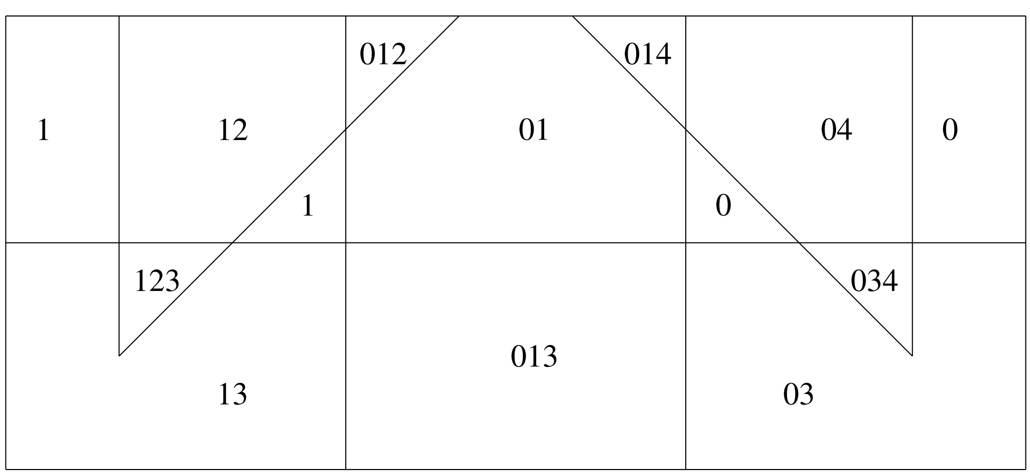
A convex realization of $\mathscr{C}12$ in $\mathbb{R}^2$.



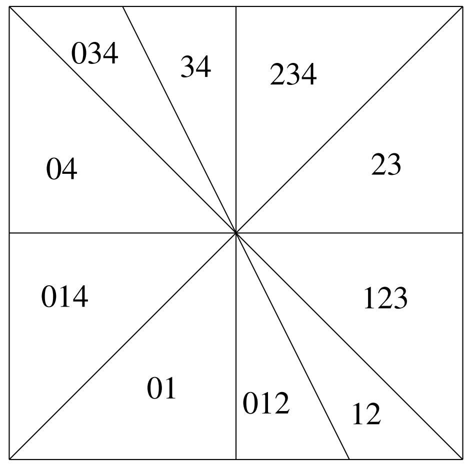
A convex realization of $\mathscr{C}13$ in $\mathbb{R}^2$.



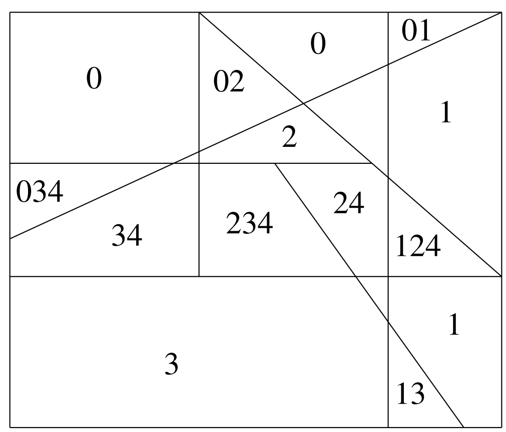
A convex realization of $\mathscr{C}14$ in $\mathbb{R}^2$.



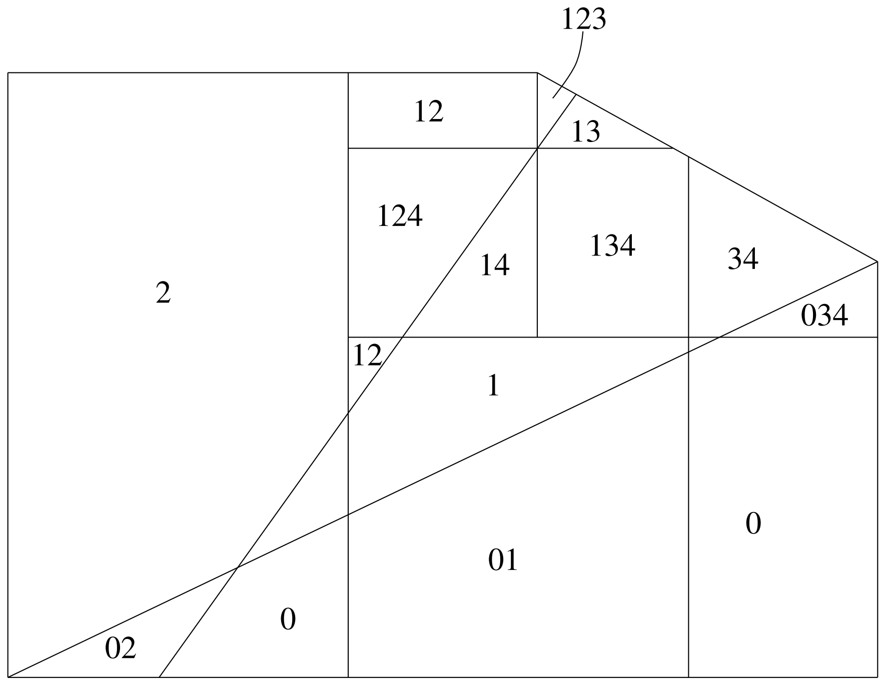
A convex realization of $\mathscr{C}15$ in $\mathbb{R}^2$.



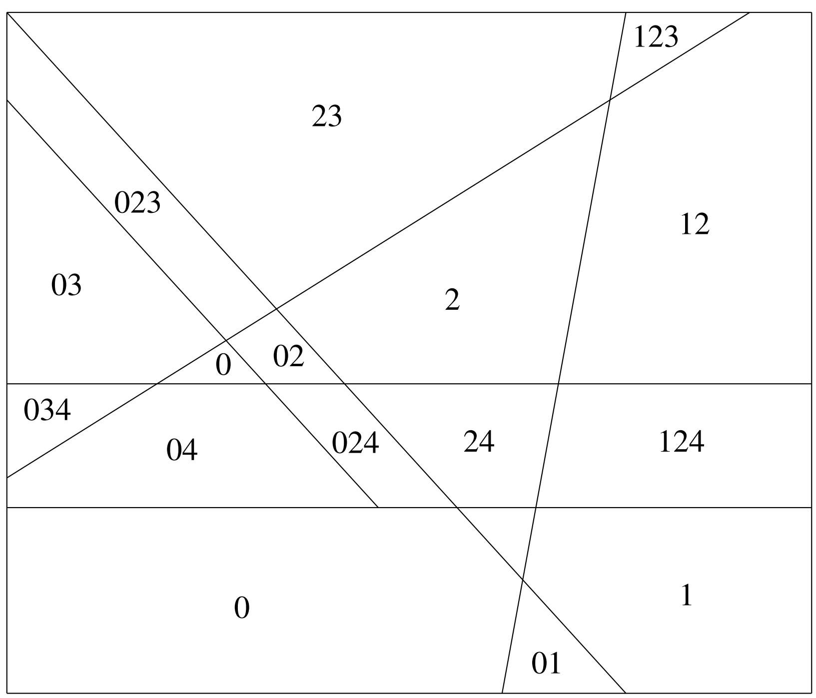
A convex realization of $\mathscr{C}16$ in $\mathbb{R}^2$.



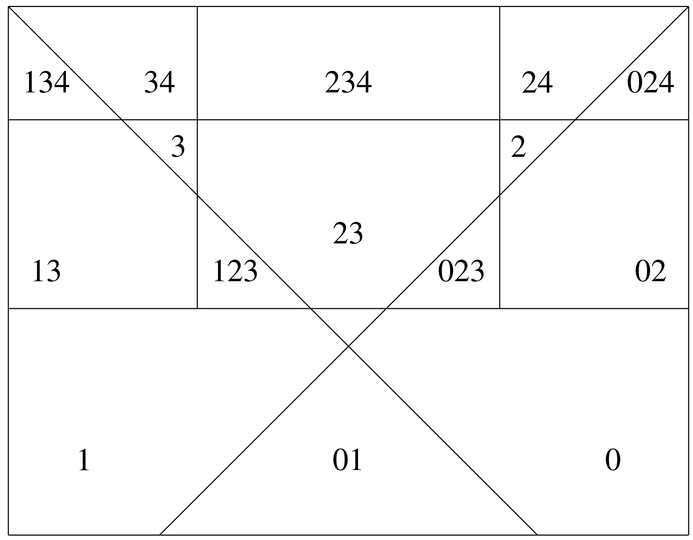
A convex realization of $\mathscr{C}17$ in $\mathbb{R}^2$.



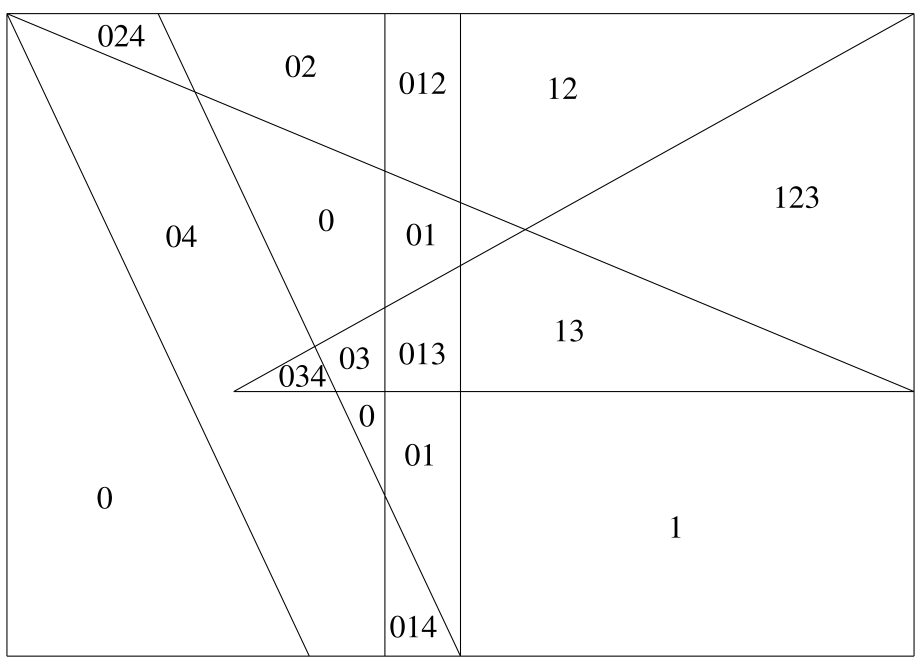
A convex realization of $\mathscr{C}18$ in $\mathbb{R}^2$.



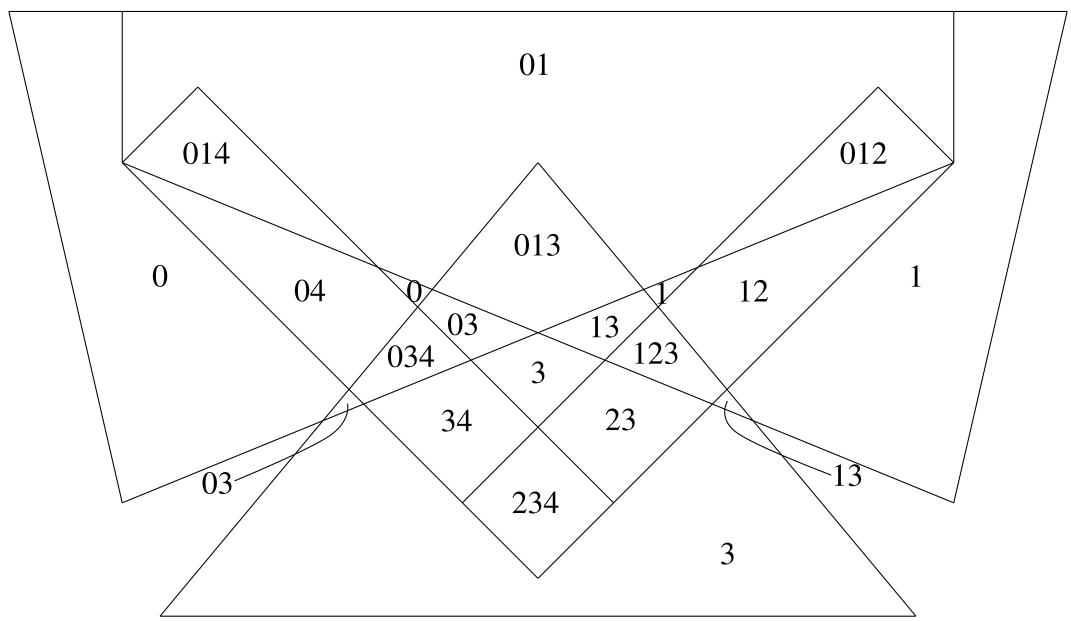
A convex realization of $\mathscr{C}19$ in $\mathbb{R}^2$.



A convex realization of $\mathscr{C}20$ in $\mathbb{R}^2$.

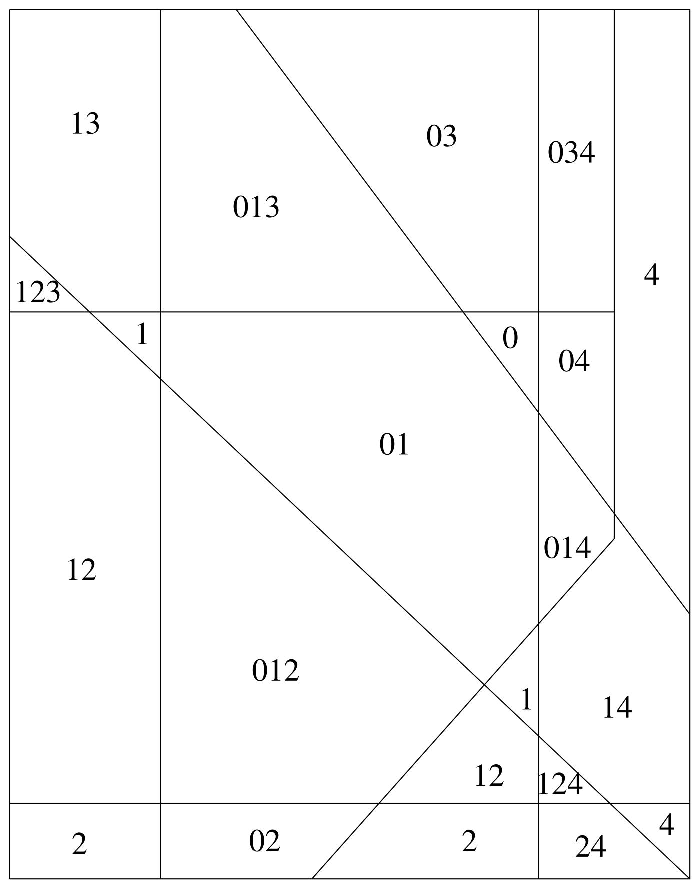


A convex realization of $\mathscr{C}21$ in $\mathbb{R}^2$.

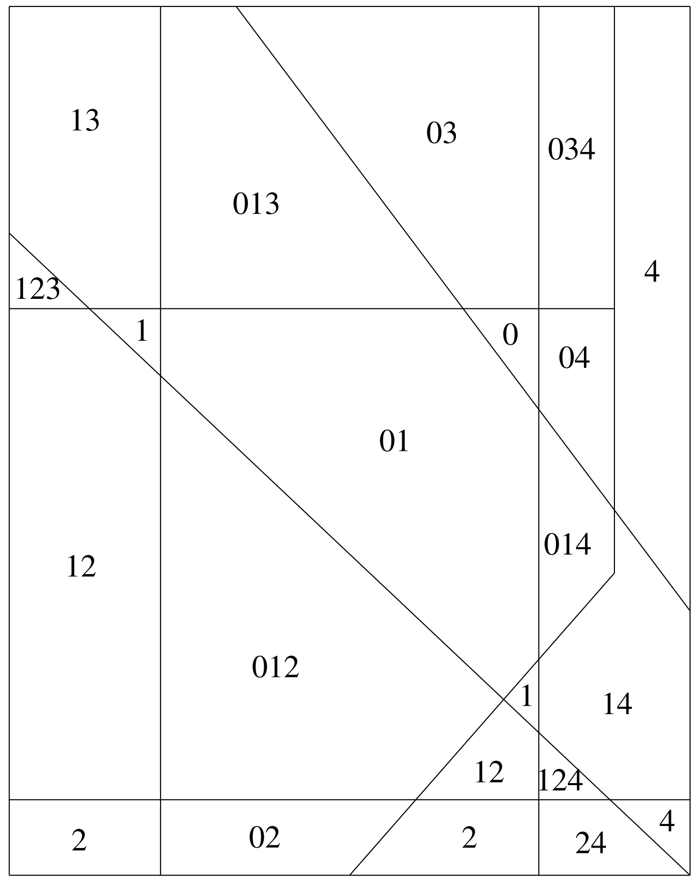


A convex realization of $\mathscr{C}22$ in $\mathbb{R}^3$. To help visualize this realization, we have drawn cross sections for planes cutting through sections at the“top” and at and near the “bottom” of the realization.

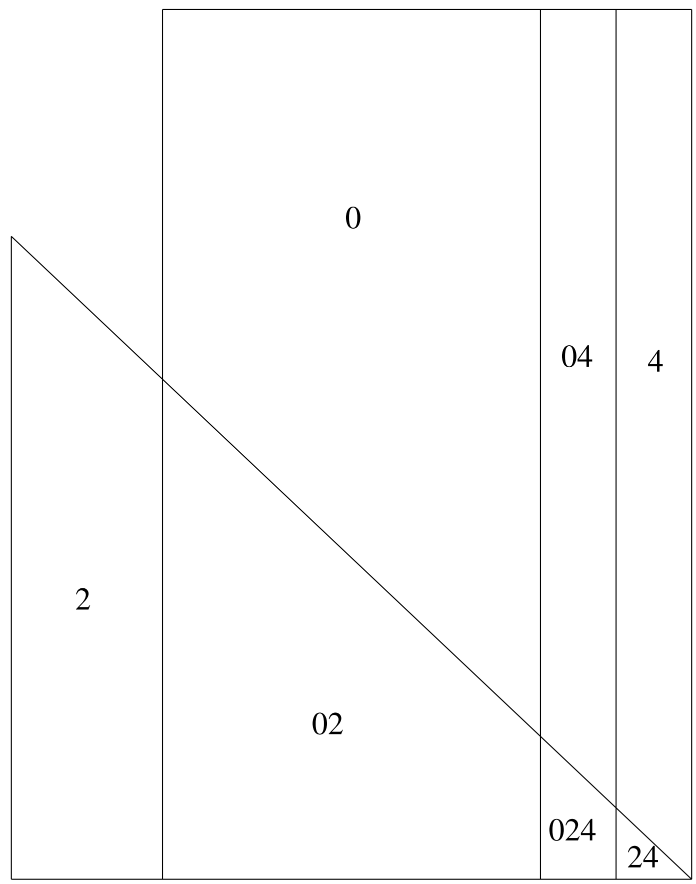
A cross section at the “bottom”:



A cross section near the “bottom”, at the “top” of sets 1 and 3:

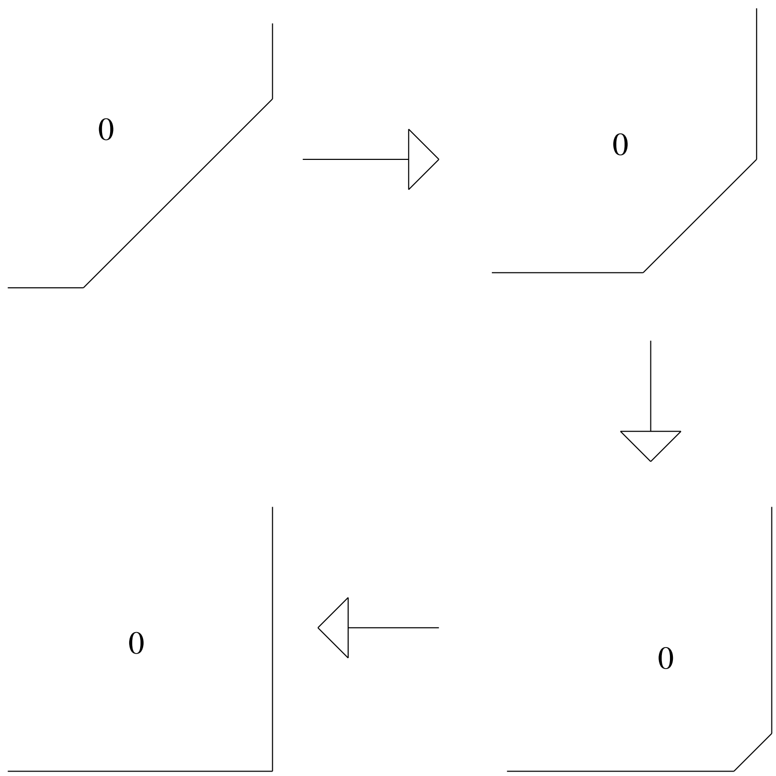


A cross section at the “top”:



To better understand this realization, it helps to think of the sets 0, 2, and 4 rising out of the page from the “bottom” cross section, and continuously morphing until they match the cross section at the “top”. In this way, the sets 2 and 4 come straight up out of the page to form a triangular prism and a rectanglular prism, respectively. As for the set 0, the cross sections coming out of the page will continuously shrink the “missing corner” on the bottom right, until the corner converges into the bottom right point of the set 0 in the “top” cross section.

The “missing corner” continuously shrinking at various cross sections. Here we are traveling upwards out of the page:



In this way we may add the codeword 024 (the only codeword missing from the diagram of the “bottom” cross section) to our realization, while avoiding the addition of the codeword 0124. Finally we note that the sets 1 and 3 only travel out of the page from the “bottom” cross section to the cross section near the “bottom”, thus forming pentagonal and rectangular prisms, respectively. From the second to third cross sections, only the sets 0, 2, and 4 remain.

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VITA

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EDUCATION

Master of Science student in Mathematics at Sam Houston State University, January 2016 – present. Thesis title: “On the Open Convexity of Neural Codes with five Neurons.”

Bachelor of Science (December 2015) in Computer Science, Sam Houston State University, Huntsville, Texas.

ACADEMIC EMPLOYMENT

Graduate Teaching Assistant, Department of Mathematics and Statistics, Sam Houston State University, January 2016 - present. Responsible for all aspects of course management, such as, creating activities to monitor learning, grading and recording grades, suggesting solutions to assigned problems, addressing student performance issues, developing ways to improve learning and understanding, assigning final grades, etc.

Research Assistant to Luís García-Puente, Department of Mathematics and Statistics, Sam Houston State University, August 2017 - May 2018. Learned technical material relevant to the research project, learned dedicated computer software, developed broad understanding of research area, helped program algorithms in related research areas.

ACADEMIC AWARDS

Outstanding Graduate Student, Department of Mathematics and Statistics, Sam Houston State University, April 2017.

Recipient of COSET Scholarship, Fall 2016 – Spring 2018

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Member of Pi Mu Epsilon

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