EXPLORING THE EQUIVALENCE BETWEEN THE POINCARE PROPERTY OF ORDER P AND THE P-NEUMANN PROPERTY IN THE VARIABLE EXPONENT SETTING

A Thesis

Presented to

The Faculty of the Department of Mathematics and Statistics

Sam Houston State University

In Partial Fulfillment

of the Requirements for the Degree of

Master of Science

---

by

Michael J. Penrod

May, 2020

EXPLORING THE EQUIVALENCE BETWEEN THE POINCARE PROPERTY OF ORDER P AND THE P-NEUMANN PROPERTY IN THE VARIABLE EXPONENT SETTING

by

Michael J. Penrod

APPROVED:

Daniel Wang, PhD
Committee Director

Damon Hay, PhD
Committee Member

Jianzhong Wang, PhD
Committee Member

John Pascarella, PhD
Dean, College of Science and Engineering Technology

DEDICATION

I have been incredibly fortunate to learn from amazing teachers and professors. At every stage of my education, from first grade through graduate school, my teachers and professors not only prepared me for the next stage, but imparted an appreciation and passion for mathematics. It is thanks to these incredible educators that I have come this far, and so I dedicate this thesis to them.

I also dedicate this thesis to my mother. My mother always encouraged me to pursue excellence, and aim high. At the same time, she created space for me to find my passion. It is thanks to her that I found my path and now pursue my journey in Mathematics.

Lastly, I dedicate this thesis to Master Allen Sharpe, Master Giselle Sharpe, and to the entire martial arts family I have known throughout my life. I learned many life skills by training with them: Integrity, concentration, perseverance, respect and obedience, self-control, humility, and indomitable spirit. These skills have prepared me in more ways than I can articulate.

ABSTRACT

Penrod, Michael J., *Exploring the Equivalence between the Poincaré Property of Order p and the p-Neumann Property in the Variable Exponent Setting*. Master of Science (Mathematics), May 2020, Sam Houston State University, Huntsville, Texas.

In [5], it was shown under weak assumptions on a matrix function Q that the Poincaré property of order p is equivalent to the p-Neumann property, where $1 < p < \infty$ is a constant exponent. We attempt to translate this result into into the variable exponent setting by replacing p with a function $p(\cdot)$. To do so, we translate the Banach function spaces L^p, and $\mathcal{L}^p_Q$, and the Sobolev space $H^{1,p}_Q$ into their variable versions, $L^{p(\cdot)}$, $\mathcal{L}^{p(\cdot)}_Q$, and $H^{1,p(\cdot)}_Q$, and investigate whether the necessary properties of these spaces still hold. We then attempt to replicate the arguments in [5], and conclude that some arguments do not translate well.

KEY WORDS: Variable Lebesgue space, Poincaré inequality, Neumann problem, Sobolev space.

ACKNOWLEDGEMENTS

Thank you to all the amazing teachers and professors that prepared me for this endeavor. Thank you to my research advisor and Analysis professor Dr. Daniel Wang for building a solid foundation and guiding me in the research process. Thank you to Dr. David Cruz-Uribe at the University of Alabama and Dr. Scott Rodney at Cape Breton University for providing open problems and useful notes on the subject of the variable spaces.

TABLE OF CONTENTS

CHAPTER 1

Introduction

In [5], an equivalence between weighted Poincaré inequalities and the existence of weak solutions to a Neumann problem related to a degenerate p-Laplacian was proven. The authors goal was to characterize the existence of a Poincaré inequality since, in many works studying regularity of elliptic PDEs, the existence of a suitable Poincaré inequality is either assumed or must be proven separately. The main result of [5] is stated below. Note that all relevant definitions can be found in chapter 2.

Theorem 1. *(Theorem 1.3 [5]) Given $1 < p < \infty$, suppose that $\gamma^{p/2} \in L^1_{loc}(E)$. The the quadratic form $\mathcal{Q}(x,\cdot)$ is p-Neumann on E if and only if $\mathcal{Q}(x,\cdot)$ has the Poincaré property of order p on E.*

In section 5 of [5], applications of this result are shown. For example, the authors describe a sufficient condition on the matrix weight Q, which illustrates when a p-Poincaré inequality holds, and consequently implies the existence of a solution to the degenerate p-Laplacian.

To prove theorem 1, the authors established useful results about the functions spaces L^p and $\mathcal{L}^p_Q$ that were then inherited by the Sobolev space $H^{1,p}_Q$. Much of the work then revolved around a mean-zero subspace of $H^{1,p}_Q$ that connects the Poincaré property and the p-Neumann property. In chapter 2, we provide a more detailed summary of how theorem 1 was proven in [5]. The authors of [5] also wish to determine whether this equivalence holds when the exponent p is replaced with a exponent function $p(\cdot)$. The variable Lebesgue space $L^{p(\cdot)}$ has many similar properties to the classical Lebesgue space L^p. The form-weighted variable Lebesgue space $\mathcal{L}^{p(\cdot)}_Q$ can also be defined similarly to $\mathcal{L}^p_Q$. This leads to us to investigate the desired equivalence between the Poincaré property of order $p(\cdot)$ and the $p(\cdot)$-Neumann property.

The function spaces mentioned before are very important in the proof of theorem 1. The classical Lebesgue space L^p consists of Lebesgue measurable functions that are scalar valued. It is then possible to modify L^p with a scalar weight function. These scalar function spaces can be extended into a space of vector valued functions with weights taking the form of matrix functions. Such a matrix-weighted space of vector valued functions is denoted $\mathcal{L}^p_Q$. It can then be shown that the Sobolev space $H^{1,p}_Q$ can be uniquely represented by pairs from $L^p \times \mathcal{L}^p_Q$. This then allows us to work with L^p and $\mathcal{L}^p_Q$ instead of directly working with elements of $H^{1,p}_Q$. To translate theorem 1 into a variable exponent setting, we develop the necessary function spaces the same way. We define a variable Lebesgue space $L^{p(\cdot)}$ as a collection of scalar valued function. We then define their weighted counterparts. We show how to extend $L^{p(\cdot)}$ to a matrix-weighted vector valued version $\mathcal{L}^{p(\cdot)}_Q$. Then, just as in the constant exponent setting, we can uniquely represent elements of the variable Sobolev space $H^{1,p(\cdot)}_Q$ by pairs from $L^{p(\cdot)} \times \mathcal{L}^{p(\cdot)}_Q$.

As stated earlier, our goal is to translate theorem 1 into the variable exponent setting. The authors of [5], Dr. David Cruz-Uribe and Dr. Scott Rodney, are the most interested is this translation. They have worked out most of the arguments to show theorem 1 holds in the variable setting, but their proofs rely on three questions. Does the conjugate norm equality (4) hold in the variable exponent setting? Is $\mathcal{L}^{p(\cdot)}_Q(E)$ a Banach space? Is $\mathcal{L}^{p(\cdot)}_Q(E)$ separable if $p_+ < \infty$, and reflexive if $1 < p_- \leq p_+ < \infty$? We will answer these questions in chapter 3 and then attempt to extend theorem 1 into the variable setting.

The remainder of this paper is organized as follows. In chapter 2, we provide definitions of variable Lebesgue spaces, the Poincaré property of order $p(\cdot)$, the $p(\cdot)$-Neumann property, as well as all other necessary definitions and established results. We also outline and summarize how the authors of [5] prove theorem 1. In chapter 3, we build the form-weighted variable Lebesgue space $\mathcal{L}_Q^{p(\cdot)}$. We also show that the properties of $\mathcal{L}_Q^p$ used in proving theorem 1, still hold in the variable version. Chapter 4 is devoted to exploring which aspects of theorem 1 translate well into the variable setting. We ultimately fail to prove or disprove the equivalence as desired.

In attempting to translate the equivalence between the Poincaré property of order p and the p-Neumann property to the variable exponent setting, we will encounter some challenges. In chapter 3, we will prove that $\mathcal{L}_Q^{p(\cdot)}(E)$ is a separable, reflexive Banach space when $1 < p_- \leq p_+ < \infty$, and then in section 4.2 verify the assumptions of Minty's Theorem. This will establish that the Poincaré property of order $p(\cdot)$ implies the existence of weak solutions to the weighted homogeneous Neumann problem. Unfortunately, we will be unable to definitively prove or disprove that weak solutions are regular. However, we will be able to show that for every $f \in L^{p(\cdot)}(v;E)$, a weak solution $(u,\mathbf{g})_f \in L^{p(\cdot)}(v;E)$ is bounded by a power of $\|f\|_{L^{p(\cdot)}(v;E)}$.

When exploring whether the $p(\cdot)$-Neumann property implies the Poincaré property of order $p(\cdot)$, we ultimately cast doubt on its validity. We will find that the desired Poincaré inequality holds for some combinations of functions and weak solutions. However, the tools we used to do this also showed possible cases where exponents appear on the norms of the inequality. Moreover, these exponents cannot be removed with the arguments presented.

This could mean that the estimation tools used in the arguments, such as theorem 18 or proposition 47 are not enough to prove that $p(\cdot)$-Neumann implies $p(\cdot)$-Poincaré, and that finer estimates exist that could prove it. However, it might be the case that there is a counterexample. In other words, there might be a measurable matrix function Q with the $p(\cdot)$-Neumann property on some bounded, open set $E \subseteq \mathbb{R}^n$ and a function $f \in C^1(\overline{E})$ with a weak solution $(u, \mathbf{g}) \in \tilde{H}_Q^{1,p(\cdot)}(v; E)$ such that the $p(\cdot)$-Poincaré inequality does not hold. If such a counter example exists, then further assumptions on the matrix function Q may be needed to establish an equivalence between the $p(\cdot)$-Neumann property and the Poincaré property of order $p(\cdot)$.

Fortunately, the work in section 4.3 suggests that a weaker result does hold. If we alter the regularity condition (23) to reflect the exponents that appear in the inequality of theorem 50, we can achieve an equivalence. This new regularity condition

$$\|u\|_{L^{p(\cdot)}(v;E)} \leq C\|f\|_{L^{p(\cdot)}(v;E)}^{\frac{r_*-1}{p_*-1}}$$

follows from the Poincaré inequality, as proven in theorem 50. By leveraging the homogeneity of the Poincaré inequality, we can prove it holds for all $C^1(\overline{E})$ functions by proving for all $f \in C^1(\overline{E})$, if $\|f\|_{L^{p(\cdot)}(v;E)} = \mu$, then

$$\|f/\mu\|_{L^{p(\cdot)}(v;E)} \leq C\|\nabla f/\mu\|_{\mathcal{L}_Q^{p(\cdot)}(E)}.$$

These details will be explored in future works.

CHAPTER 2

Motivation

2.1 Preliminaries

We begin by introducing some notation. Throughout the rest of this paper, let $E \subseteq \mathbb{R}^n$ be a fixed domain. We will sometimes require that E is bounded and open. In such cases, this will be explicitly stated for clarity. Let $\mathcal{S}_n$ denote the collection of all positive semi-definite, $n \times n$ self-adjoint matrices. Recall that an $n \times n$ matrix Q, with real valued entries q_{ij}, is positive semi-definite if for all nonzero $\xi \in \mathbb{R}^n$, $\xi^T Q \xi \geq 0$, and self-adjoint if for all $1 \leq i, j \leq n$, $q_{ij} = q_{ji}$. We can now define matrix functions. Let $Q : E \to \mathcal{S}_n$ be a matrix with Lebesgue measurable functions as entries, that is for $x \in E$,

$$Q(x) = \begin{bmatrix} q_{11}(x) & q_{12}(x) & \cdots & q_{1n}(x) \\ q_{21}(x) & q_{22}(x) & \cdots & q_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1}(x) & q_{n2}(x) & \cdots & q_{nn}(x) \end{bmatrix}$$

where for all $1 \leq i, j \leq n$, $q_{ij} : E \to \mathbb{R}$ is Lebesgue measurable. Define the associated quadratic form $\mathcal{Q}(x, \xi) = \xi^T Q(x) \xi$, $x \in E$ a.e. and $\xi \in \mathbb{R}^n$. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^n$. We define $\gamma(x) = |Q(x)|_{op} = \sup_{|\xi|=1} |Q(x)\xi|$ to be the operator norm of $Q(x)$. Let v be a weight on E, i.e. v is a non-negative function in $L^1_{\text{loc}}(E)$ with $v(x) < \infty$ for almost every $x \in E$. Given a Lebesgue measurable function f, we define the weighted average of f on E by

$$f_E = f_{E,v} = \frac{1}{v(E)} \int_E f(x) v(x) dx$$

where $v(E)$ is the weighted measure of the E, that is, $v(E) = \int_E v(x)dx$.

We now define the Poincaré property of order p and the p-Neumann property. The variable versions of these definitions can be found in section 4.1

Definition 2. Given $1 \le p < \infty$, a quadratic form $\mathcal{Q}$ is said to have the Poincaré property of order p on E if there is a positive constant $C_p = C_p(E)$ such that for all $f \in C^1(\overline{E})$,

$$\begin{aligned} \int_E |f(x) - f_E|^p v(x)\, dx &\le C_p \int_E \left| \sqrt{Q(x)} \nabla f(x) \right|^p dx \\ &= C_p \int_E \mathcal{Q}(x, \nabla f(x))^{p/2}\, dx. \end{aligned} \tag{1}$$

Definition 3. Given $1 \le p < \infty$, a quadratic form $\mathcal{Q}$ is said to have the p-Neumann property on E if the following hold:

1. Given any $f \in L^p(v;E)$, there exists a weak solution $(u,\mathbf{g})_f \in \tilde{H}_Q^{1,p}(v;E)$ to the weighted homogeneous Neumann problem

$$\begin{cases} \operatorname{div}\left(\left| \sqrt{Q(x)} \nabla u(x) \right|^{p-2} Q(x) \nabla u(x) \right) & = |f(x)|^{p-2} f(x) v(x) \text{ in } E \\ \mathbf{n}^t \cdot Q(x) \nabla u & = 0 \text{ on } \partial E, \end{cases} \tag{2}$$

   where $\mathbf{n}$ is the outward unit normal vector of ∂E.

2. Any weak solution $(u,\mathbf{g})_f \in \tilde{H}_Q^{1,p}(E)$ of (2) is regular: that is, there is a positive constant $C_p = C_p(v,E)$ such that

$$\|u\|_{L^p(v;E)} \le C_p \|f\|_{L^p(v;E)}. \tag{3}$$

In the following section, we provide the definition of the degenerate Sobolev space $H_Q^{1,p}(v;E)$ and the precise definition of weak solutions. We also outline the methodology used in [5] to prove theorem 1, which we introduced in chapter 1.

2.2 Established Methodology

The proof of theorem 1 relies on three important facts. The first is that the form-weighted vector valued Lebesgue space, which we define in chapter 3, is a Banach space. The second is that if $\gamma^{p/2} \in L^1_{\text{loc}}(E)$, then this space is separable if $p < \infty$, and additionally reflexive if $1 < p < \infty$. The third is the following equality between conjugate exponent norms. For $1 \leq p \leq \infty$, let p' be the conjugate norm, that is $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have for $1 \leq p < \infty$

$$\|f\|_p^{p-1} = \||f|^{p-1}\|_{p'} \tag{4}$$

We will discuss how these three facts are used in [5] to prove the equivalence between the Poincaré property of order p and the p-Neumann property. The variable exponent versions of these three properties are proven in chapter 3. We first introduce the classical Lebesgue space L^p, as well as the form-weighted Lebesgue space $\mathcal{L}^p_Q$.

Definition 4. Given $1 \leq p \leq \infty$, a set E, and a weight v on E, we define the weighted Lebesgue space, $L^p(v;E)$ to be the collection of Lebesgue measurable functions $f : E \to \mathbb{R}$ satisfying $\|f\|_{L^p(v;E)} < \infty$, where

$$\|f\|_{L^p(v;E)} = \begin{cases} \left(\int_E |f(x)|^p v(x)dx\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \operatorname{esssup}_{x \in E} |f(x)| & \text{if } p = \infty \end{cases}$$

If $v(x) = 1$ for a.e. $x \in E$, this space is the classical Lebesgue space $L^p(E)$.

Note that in this definition, f is a scalar-valued function. We can also extend this to vector-valued functions $\mathbf{f}$. In this vector valued setting, we can replace the scalar-valued weight v with a matrix valued function Q. This leads to the form-weighted vector-valued Lebesgue space $\mathcal{L}^p_Q(E)$, which we now define.

Definition 5. Given $1 \leq p < \infty$, a bounded, open set $E \subseteq \mathbb{R}^n$, and a matrix function $Q : E \to \mathcal{S}_n$, define the form-weighted vector-valued Lebesgue space, $\mathcal{L}^p_Q(E)$, to be the collection of all $\mathbb{R}^n$ valued functions $\mathbf{f} : E \to \mathbb{R}^n$ satisfying

$$\|\mathbf{f}\|_{\mathcal{L}^p_Q(E)} = \left(\int_E \mathcal{Q}(x, \mathbf{f}(x))^{p/2} dx \right)^{1/p} = \left(\int_E \left| \sqrt{Q(x)} \mathbf{f}(x) \right|^p dx \right)^{1/p} < \infty$$

A key connection between these spaces lies in their norms and the eigenvalues of the matrix function Q. For $\mathbf{f} \in \mathcal{L}^p_Q(E)$, we can rewrite $|\sqrt{Q(x)}\mathbf{f}(x)|$ as the $\sum_{j=1}^n |\tilde{f}_j(x)|^2 \lambda_j(x)$, where each $\lambda_j(x)$ is an eigenvalue of $Q(x)$ for a.e. $x \in E$. Consequently, we can prove that the $\mathcal{L}^p_Q$ norm is equivalent to the sum of $L^p(\lambda_j^{p/2}; E)$ norms. Since weighted Lebesgue spaces are complete, this equivalence implies that $\mathcal{L}^p_Q(E)$ is complete in its norm. These facts are proven true for the variable exponent setting in chapter 3. The completeness of $\mathcal{L}^p_Q$ and L^p is inherited by the Sobolev space $H^{1,p}_Q(v; E)$, which we now define as a collection of equivalence classes of Cauchy sequences of $C^1(\overline{E})$ functions.

Definition 6. For $1 \leq p < \infty$, the Sobolev space $H^{1,p}_Q(v; E)$ is the abstract completion of $C^1(\overline{E})$ with respect to the norm

$$\|f\|_{H^{1,p}_Q(v;E)} = \|f\|_{L^p(v;E)} + \|\nabla f\|_{\mathcal{L}^p_Q(E)}.$$

It is known that $L^p(v; E)$ is complete. It is then shown in [5] that $\mathcal{L}^p_Q(E)$ is complete.

From this, it can then be shown that $H^{1,p}_Q(v;E)$ is isometrically isomorphic to a closed subspace of $L^p(v;E)\times\mathcal{L}^p_Q(E)$, and hence is complete. However, because of the degeneracy of Q, we cannot represent $H^{1,p}_Q(v;E)$ as a space of functions except in special situations. Instead, we use a unique pair $\mathbf{f}=(u,\mathbf{g})\in L^p(v;E)\times\mathcal{L}^p_Q(E)$ to represent the elements of $H^{1,p}_Q(v;E)$. Thus we refer to elements of $H^{1,p}_Q(v;E)$ by their representative pairs. However, the vector $\mathbf{g}$ need not be uniquely determined by u. See [1–3, 6, 8–10, 12, 13] for more information. However, if $f\in C^1(\overline{E})\cap H^{1,p}_Q(v;E)$, of more simply, if $f\in C^1(\overline{E})$ and $\gamma^{p/2}\in L^1_{\text{loc}}(E)$, then $(f,\nabla f)\in H^{1,p}_Q(v;E)$.

To use this space to prove theorem 1, it is important that we restrict our attention to the "mean zero" subspace of $H^{1,p}_Q(v;E)$, which is defined by

$$\tilde{H}^{1,p}_Q(v;E)=\{(u,\mathbf{g})\in H^{1,p}_Q(v;E):\int_E u(x)v(x)dx=0\}$$

It can be shown that $\tilde{H}^{1,p}_Q(v;E)$ is a closed subspace of $H^{1,p}_Q(v;E)$. Since $H^{1,p}_Q(v;E)$ is complete, $\tilde{H}^{1,p}_Q(v;E)$ is as well. The completeness of $\tilde{H}^{1,p}_Q(v;E)$ is then leveraged in [5] to prove $C^1(\overline{E})\cap\tilde{H}^{1,p}_Q(v;E)$ is dense in $\tilde{H}^{1,p}_Q(v;E)$. This result allows us to define weak solutions to the weighted homogeneous Neumann problem (2). Given $f\in L^p(v;E)$, we say that a pair $(u,\mathbf{g})_f\in\tilde{H}^{1,p}_Q(v;E)$ is a weak solution of (2) if for all test functions $\varphi\in C^1(\overline{E})\cap\tilde{H}^{1,p}_Q(v;E)$,

$$\int_E|\sqrt{Q(x)}\mathbf{g}(x)|^{p-2}(\nabla\varphi(x))^TQ(x)\mathbf{g}(x)dx=-\int_E|f(x)|^{p-2}f(x)\varphi(x)v(x)dx \tag{5}$$

It should be noted that since $C^1(\overline{E})\cap\tilde{H}^{1,p}_Q(v;E)$ is dense in $\tilde{H}^{1,p}_Q(v;E)$, an approximation argument shows that this definition holds for all $(u,\mathbf{g})\in\tilde{H}^{1,p}_Q(v;E)$. That is, we can replace φ with u, and $\nabla\varphi$ with $\mathbf{g}$. With this definition of weak solutions, the following lemma is proven.

Lemma 7. *(Lemma 3.1 in [5]) Given* $1 < p < \infty$ *and* $f \in L^p(v;E)$*, if* $(u,\mathbf{g})_f \in \tilde{H}^{1,p}_Q(v;E)$ *is a weak solution of the Neumann problem* (2)*, then* $\|(u,\mathbf{g})\|_{H^{1,p}_Q(v;E)} \lesssim \|f\|_{L^p(v;E)}$ *if and only if* $\|u\|_{L^p(v;E)} \lesssim \|f\|_{L^p(v;E)}$.

The forward direction of theorem 1 then follows from this lemma. In summary, the completeness of $\mathcal{L}^p_Q(E)$ ultimately proves that $C^1(\overline{E}) \cap \tilde{H}^{1,p}_Q(v;E)$ is dense in $\tilde{H}^{1,p}_Q(v;E)$. Using this dense subset, lemma 7 is proven, from which it is shown that if $\gamma^{p/2} \in L^1_{\text{loc}}(E)$ then p-Neumann implies the Poincaré property of order p.

To prove the reverse direction, Minty's theorem from [14] is used to prove existence of weak solutions. To state Minty's theorem, we first establish some notation. Given a reflexive Banach space $\mathscr{B}$, denote its dual space by $\mathscr{B}^*$. Given a functional $\alpha \in \mathscr{B}^*$, write its value at $\varphi \in \mathscr{B}$ as $\alpha(\varphi) = \langle \alpha, \varphi \rangle$. Thus, if $\beta : \mathscr{B} \to \mathscr{B}^*$ and $u \in \mathscr{B}$, then we have $\beta(u) \in \mathscr{B}^*$ and so its value at φ is denoted by $\beta(u)(\varphi) = \langle \beta(u), \varphi \rangle$. We now state Minty's theorem.

Theorem 8. *(Minty, Theorem 4.1 in [5])Let* $\mathscr{B}$ *be a reflexive, separable Banach space and fix* $\Gamma \in \mathscr{B}^*$*. Suppose that* $\mathfrak{T} : \mathscr{B} \to \mathscr{B}^*$ *is a bounded operator that is:*

1. *Monotone:* $\langle \mathfrak{T}(u) - \mathfrak{T}(\varphi), u - \varphi \rangle \geq 0$ *for all* $u, \varphi \in \mathscr{B}$*;*

2. *Hemicontinuous: for* $z \in \mathbb{R}$*, the mapping* $z \mapsto \langle \mathfrak{T}(u + z\varphi), \varphi \rangle$ *is continuous for all* $u, \varphi \in \mathscr{B}$*;*

3. *Almost Coercive: there exists a constant* $\lambda > 0$ *so that* $\langle \mathfrak{T}(u), u \rangle > \langle \Gamma, u \rangle$ *for any* $u \in \mathscr{B}$ *satisfying* $\|u\|_{\mathscr{B}} > \lambda$.

Then the set of $u \in \mathscr{B}$ such that $\mathfrak{T}(u) = \Gamma$ is non-empty.

To apply Minty's theorem in the context of theorem 1, it must be shown that if $\gamma^{p/2} \in L^1_{\mathrm{loc}}(E)$, then $\tilde{H}^{1,p}_Q(v;E)$ is separable if $p < \infty$ and reflexive when $1 < p < \infty$. This in turn depends on showing $\mathcal{L}^p_Q(E)$ is separable if $p < \infty$ and reflexive for $1 < p < \infty$ when $\gamma^{p/2} \in L^1_{\mathrm{loc}}(E)$. Once this is shown, we let $\mathscr{B} = \tilde{H}^{1,p}_Q(v;E)$ and define the operator $\mathfrak{T} : \tilde{H}^{1,p}_Q(v;E) \to \left(\tilde{H}^{1,p}_Q(v;E)\right)^*$ for Minty's theorem as follows. For $\mathbf{u} = (u, \mathbf{g})$ and $\mathbf{w} = (w, \mathbf{h})$ in $\tilde{H}^{1,p}_Q(v;E)$, $\mathfrak{T}(\mathbf{u})$ acts on $\mathbf{w}$ by computing the left hand side of our definition of weak solutions (5), that is,

$$\langle \mathfrak{T}(\mathbf{u}), \mathbf{w} \rangle = \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p-2}(\mathbf{h}(x))^T Q(x)\mathbf{g}(x)dx.$$

Now define $\Gamma_f = \Gamma \in \left(\tilde{H}^{1,p}_Q(v;E)\right)^*$ as the right hand side of our definition of weak solutions (5), that is for any $f \in L^p(v;E)$, Γ acts on $\mathbf{w} = (w, \mathbf{h}) \in \tilde{H}^{1,p}_Q(v;E)$ by

$$\Gamma(\mathbf{w}) = \langle \Gamma, \mathbf{w} \rangle = -\int_E |f(x)|^{p-2} f(x) w(x) v(x) dx.$$

Note that $\Gamma_f = \Gamma$ is dependent on $f \in L^p(v;E)$. For simplicity, we will simply write Γ when f is clear from the context. To see why the operator $\mathfrak{T}$ satisfies the assumptions of Minty's theorem, see theorems 4.4 through 4.7 in [5]. The variable versions of these operators and proofs can be found in section 4.2.

In the proofs of theorems 4.4 through 4.7 in [5], Holder's inequality is used, followed by the conjugate norm equality (4). As an example, we provide the proof that for $1 \le p < \infty$,

$\mathcal{T}$ is bounded on $\tilde{H}^{1,p}_Q(v;E)$. Fix $\mathbf{u}=(u,\mathbf{g})$ and $\mathbf{w}=(w,\mathbf{h})$in $\tilde{H}^{1,p}_Q(v;E)$. By the Cauchy-Schwartz inequality,

$$|\mathbf{h}(x)^T Q(x)\mathbf{g}(x)| \leq |\sqrt{Q(x)}\mathbf{h}(x)||\sqrt{Q(x)}\mathbf{g}(x)|$$

Applying this to the integrand of the operator $\mathcal{T}$ and then using Holder's inequality, we get

$$|\langle \mathcal{T}(\mathbf{u}),\mathbf{w}\rangle| \leq \left\| |\sqrt{Q(x)}\mathbf{g}(x)|^{p-1}\right\|_{L^{p'}(v;E)} \left\| |\sqrt{Q(x)}\mathbf{h}(x)|\right\|_{L^{p}(v;E)}.$$

Now by our conjugate equality, (4), we get

$$|\langle \mathcal{T}(\mathbf{u}),\mathbf{w}\rangle| \leq \left\| |\sqrt{Q(x)}\mathbf{g}(x)|\right\|^{p-1}_{L^{p}(v;E)} \left\| |\sqrt{Q(x)}\mathbf{h}(x)|\right\|_{L^{p}(v;E)} = \|\mathbf{g}\|^{p-1}_{\mathcal{L}^p_Q(E)}\|\mathbf{h}\|_{\mathcal{L}^p_Q(E)}.$$

From the definitions of the $\mathcal{L}^p_Q$ and $H^{1,p}_Q$ norms, $\|\mathbf{g}\|^{p-1}_{\mathcal{L}^p_Q(E)}\|\mathbf{h}\|_{\mathcal{L}^p_Q(E)}$ is bounded by $\|\mathbf{u}\|^{p-1}_{H^{1,p}_Q(v;E)}\|\mathbf{w}\|_{H^{1,p}_Q(v;E)}$, which proves that $\mathcal{T}$ is bounded. This process of using Holder's inequality followed by (4) is heavily used in many of the arguments in [5].

2.3 Objective

Our objective for this thesis is to determine whether theorem 1, which is the main result of [5], holds when we replace the constant exponent p with an exponent function, $p(\cdot)$. However, before attempting to answer this question, we first introduce how changing the constant exponent to an exponent function affects the definition of Lebesgue spaces. We begin by defining what exponent functions are, and introduce some notation. Then we will define variable Lebesgue spaces.

Definition 9. Given a set $E \subseteq \mathbb{R}^n$, an exponent function is a Lebesgue measurable function

$p(\cdot) : E \to [1,\infty]$. Denote the collection of all exponent functions on E by $\mathscr{P}(E)$. Define the set E_∞ by $E_\infty = \{x \in E : p(x) = \infty\}$. Define $p_-(E) = p_- = \operatorname{essinf}_{x\in E}(p(x))$ and $p_+(E) = p_+ = \operatorname{esssup}_{x\in E}(p(x))$.

Definition 10. Given $E \subseteq \mathbb{R}^n$, $p(\cdot) \in \mathscr{P}(E)$ and a Lebesgue measurable function f, define the modular functional (or simply "modular") associated with $p(\cdot)$ by

$$\rho_{p(\cdot),E} = \int_{E\setminus E_\infty} |f(x)|^{p(x)}dx + \|f\|_{L^\infty(E_\infty}.$$

If f is unbounded on E_∞ or $f(\cdot)^{p(\cdot)} \notin L^1(E\setminus E_\infty)$ then we define $\rho_{p(\cdot),E} = +\infty$. When $|E_\infty| = 0$ we let $\|f\|_{L^\infty(E_\infty)} = 0$; when $|E\setminus E_\infty| = 0$, then $\rho_{p(\cdot),E}(f) = \|f\|_{L^\infty(E_\infty)}$. In situations where there is no ambiguity we will simply write $\rho_{p(\cdot)}(f)$ or $\rho(f)$.

Definition 11. Let $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$. Let v be a weight on E, i.e. a non-negative measurable function in $L^1_{\text{loc}}(E)$ with $v(x) < \infty$ for a.e. $x \in E$.

1. We define the variable Lebesgue space $L^{p(\cdot)}(E)$ to be the collection of all Lebesgue measurable functions $f : E \to \mathbb{R}$ satisfying

$$\|f\|_{p(\cdot)} = \inf\{\mu > 0 : \rho\left(\frac{f}{\mu}\right) \leq 1\} < \infty.$$

2. We define the weighted variable Lebesgue space $L^{p(\cdot)}(v;E)$ to be the collection of all Lebesgue measurable functions satisfying

$$\|f\|_{L^{p(\cdot)}(v;E)} = \|fv\|_{p(\cdot)} < \infty$$

To illustrate how the norm $\|\cdot\|_{p(\cdot)}$ is different from the classical norm $\|\cdot\|_p$, we consider an example. Let $E=(1,\infty)$ and $p(x)=x$ for all $x\in E$. Let $f(x)=1$. Since E has infinite measure and f is a constant function, $\|f\|_p=(\int_E |f(x)|^p dx)^{1/p}=\infty$ for all $1\leq p<\infty$. However, if we let $\mu=2$, then

$$\rho\left(\frac{f}{\mu}\right)=\int_1^{\infty}2^{-x}dx=\frac{1}{2\ln 2}<1$$

Thus $\|f\|_{p(\cdot)}\leq 2$. This simple example illustrates a difference between classical Lebesgue spaces and variable Lebesgue spaces. However, there are some similarities. In the classical setting, we have many well known results, such as Fatou's lemma, Holder's inequality, and the dominated convergence theorem. We also have that $L^p(E)$ is a Banach Space. There are analogous results in $L^{p(\cdot)}(E)$ that we now introduce. The proofs of these results can be found in [4].

Theorem 12. *(Fatou's lemma in $L^{p(\cdot)}(E)$, Theorem 2.61 in [4]) Given $E\subseteq\mathbb{R}^n$ and $p(\cdot)\in\mathscr{P}(E)$, suppose the sequence $\{f_k\}_{k=1}^{\infty}\subseteq L^{p(\cdot)}(E)$ is such that $f_k\to f$ pointwise almost everywhere. If $\liminf_{k\to\infty}\|f_k\|_{p(\cdot)}<\infty$, then $f\in L^{p(\cdot)}(E)$ and $\|f\|_{p(\cdot)}\leq\liminf_{k\to\infty}\|f_k\|_{p(\cdot)}$.*

Note that Fatou's lemma translates unchanged into the variable setting. A version of Holder's inequality also holds in $L^{p(\cdot)}(E)$, but a constant dependent on $p(\cdot)$ must be inserted. Before stating this version of Holder's inequality, we introduce some notation. Let $E_1=\{x\in E: p(x)=1\}$ and $E_*=\{x\in E: 1<p(x)<\infty\}$. We say $p(\cdot)$ and $p'(\cdot)$ are conjugate exponent functions on $E\subseteq\mathbb{R}^n$ if for almost every $x\in E$,

$$\frac{1}{p(x)}+\frac{1}{p'(x)}=1$$

In defining conjugate exponents, we adopt the convention that $1/\infty = 0$. We now state the modified Holder's inequality.

Theorem 13. *(Holder's inequality in $L^{p(\cdot)}(E)$, Theorem 2.26 in [4]) Given a set $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$, for all $f \in L^{p(\cdot)}(E)$ and $g \in L^{p'(\cdot)}(E)$, $fg \in L^1(E)$ and*

$$\int_E |f(x)g(x)|dx \leq K_{p(\cdot)} \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)},$$

where

$$K_{p(\cdot)} = \left(\frac{1}{p_-} - \frac{1}{p_+} + 1\right) \|\chi_{E_*}\|_\infty + \|\chi_{E_\infty}\|_\infty + \|\chi_{E_1}\|_\infty$$

Note that if $p(\cdot)$ is a constant function, this inequality becomes the classical Holder's inequality with $K_{p(\cdot)} = 1$. However, if E_1, E_*, and E_∞ have positive measure, then $K_{p(\cdot)} = 4$. Hence, for all $p(\cdot) \in \mathscr{P}(E)$, $1 \leq K_{p(\cdot)} \leq 4$, and so this version of Holder's inequality is still useful. However, the dominated convergence theorem does not translate as well into variable Lebesgue spaces.

Theorem 14. *(Dominated convergence theorem in $L^{p(\cdot)}(E)$, Proposition 2.67 in [4]) Given $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$, suppose $p_+ < \infty$. If the sequence $\{f_k\}_{k=1}^\infty$ is such that $f_k \to f$ pointwise almost everywhere, and there exists $g \in L^{p(\cdot)}(E)$ such that $|f_k(x)| \leq g(x)$ almost everywhere, then $f \in L^{p(\cdot)}(E)$ and $\|f - f_k\|_{p(\cdot)} \to 0$ as $k \to \infty$. Moreover, if $p_+ = \infty$, then this result is always false.*

While the dominated convergence theorem requires an additional assumption to hold in $L^{p(\cdot)}(E)$, completeness does not. Its well known that for $1 \leq p \leq \infty$, $L^p(E)$ is a Banach space. This also holds true for $L^{p(\cdot)}(E)$ as well.

Theorem 15. *(Theorem 2.71 in [4]) Given* $E \subseteq \mathbb{R}^n$ *and* $p(\cdot) \in \mathscr{P}(E)$*,* $L^{p(\cdot)}(E)$ *is a Banach space.*

With these results, we see some similarities and differences between the classical Lebesgue spaces and variable Lebesgue spaces. In the following chapter, we attempt to extend the methodology used in section 2.2 to this variable exponent setting.

CHAPTER 3

Main Results on $\mathcal{L}_Q^{p(\cdot)}$

3.1 Equality of Conjugate Norms

In this section, we state and prove the variable versions of the three questions introduced at the start of section 2.2. We begin by determining the validity of the conjugate norm equation (4) in the variable exponent setting. When we replace p with $p(\cdot)$ and p' with $p'(\cdot)$, the conjugate norm equation becomes

$$\|f\|_{p(\cdot)}^{p(\cdot)-1} = \||f|^{p(\cdot)-1}\|_{p'(\cdot)}.$$

Upon careful examination of this statement, we see a problem with notation. On the right hand side, writing the exponent function as $p(\cdot)-1$ inside the $p'(\cdot)$ norm makes sense as the definition of the $p'(\cdot)$ norm involves an integral of $|f(x)|^{p(x)-1}$, with x varying over the entire domain. However on the left hand side, writing $p(\cdot)-1$ outside the norm makes no sense. Outside the norm, $p(\cdot)$ is not varying over the domain. Replacing $p(\cdot)-1$ with $p(x)-1$ makes no sense either, since the norms on both sides of the statement should yield a constant, not a function of x. However, a qualitative result from (4) in the constant exponent setting is that the quantities $\|f\|_p$ and $\||f|^{p-1}\|_{p'}$ are always comparable. In other words, $\|f\|_p$ is finite if and only if $\||f|^{p-1}\|_{p'}$ is finite, and there is a relationship between them. With this observation, we can ask if the same is true in the variable exponent setting. Unfortunately, the answer depends on the exponent function itself.

Theorem 16. *Let $E \subseteq \mathbb{R}^n$. Let $p(\cdot) \in \mathscr{P}(E)$ such that $p_+(E \backslash E_\infty) = \infty$. Then there exists a measurable function $f : E \to \mathbb{R}$ such that*

$$\|f\|_{p(\cdot)} < \infty \tag{6}$$

$$\||f|^{p(\cdot)-1}\|_{p'(\cdot)} = \infty \tag{7}$$

Proof. Let $E \subseteq \mathbb{R}^n$. Let $p(\cdot) \in \mathscr{P}(E)$ with $p_+(E \backslash E_\infty) = \infty$. By the definition of the essential supremum, there exists a sequence of sets $\{E_k\}_{k=1}^\infty$ with finite measure such that

1. if $x \in E_k$, then $p(x) \geq k$,

2. $E_k \subseteq E \backslash E_\infty$,

3. $E_{k+1} \subseteq E_k$ and $|E_k \backslash E_{k+1}| > 0$, and

4. $|E_k| \to 0$ as $k \to \infty$.

For each k, define $I_k = E_k \backslash E_{k+1}$. Define $f : E \to \mathbb{R}^n$ by for all $x \in E$,

$$f(x) = \left(\sum_{k=1}^\infty \frac{1}{|I_k|} \chi_{I_k}(x) \right)^{1/p(x)}.$$

To prove (6) consider $\mu = 2$. Then

$$\begin{aligned}
\rho_{p(\cdot)}(f/2) &= \int_{E \backslash E_\infty} 2^{-p(x)} \sum_{k=1}^\infty \frac{1}{|I_k|} \chi_{I_k}(x) dx && (f(x) = 0 \text{ on } E_\infty) \\
&= \sum_{k=1}^\infty \frac{1}{|I_k|} \int_{I_k} 2^{-p(x)} dx \\
&\leq \sum_{k=1}^\infty \frac{1}{|I_k|} \int_{I_k} 2^{-k} dx && (2 > 1)
\end{aligned}$$

Since 2^{-k} is constant with respect to x, this last expression simplifies to a geometric series equal to 1. Thus, $\|f\|_{p(\cdot)} = \inf\{\mu > 0 : \rho_{p(\cdot)}(f/\mu) \le 1\} \le 2$ which is finite. To prove (7) let $\mu > 0$. Observe that as $k \to \infty$, $p_-(I_k) \to \infty$ and $p'_+(I_k) \to 1$. Thus there exists $K \in \mathbb{N}$ such that for all $k \ge K$, if $x \in I_k$,

$$\mu^{-p'(x)} > \frac{1}{2}\mu^{-1} \tag{8}$$

Note that $\{x \in E : p'(x) = \infty\} = \{x \in E : p(x) = 1\} = E_1$. This is reflected below in the domain of the integral in the modular $\rho_{p'(\cdot)}$. Now observe that

$$\begin{aligned}
\rho_{p'(\cdot)}\left(\frac{|f|^{p(\cdot)-1}}{\mu}\right) &= \int_{E\setminus E_1} \mu^{-p'(x)} \left(\sum_{k=1}^{\infty} \frac{1}{|I_k|}\chi_{I_k}(x)\right)^{p'(x)\frac{(p(x)-1)}{p(x)}} dx \\
&= \int_{E\setminus E_1} \mu^{-p'(x)} \sum_{k=1}^{\infty} \frac{1}{|I_k|}\chi_{I_k}(x) && \left(\frac{p(x)-1}{p(x)} = \frac{1}{p'(x)}\right) \\
&= \sum_{k=1}^{\infty} \frac{1}{|I_k|} \int_{I_k} \mu^{-p'(x)} dx \\
&\ge \sum_{k=K}^{\infty} \frac{1}{|I_k|} \int_{I_k} \mu^{-p'(x)} dx && \text{(remove terms)} \\
&\ge \sum_{k=K}^{\infty} \frac{1}{|I_k|} \int_{I_k} \frac{1}{2}\mu^{-1} dx && \text{(see (8))} \\
&= \sum_{k=K}^{\infty} \frac{1}{2}\mu^{-1} \\
&= \infty
\end{aligned}$$

Thus $\||f|^{p(\cdot)-1}\|_{p'(\cdot)} = \infty$. □

The key assumption in this counter example is that the exponent function is unbounded on $E\setminus E_\infty$. However, if we assume the exponent function is bounded on $E\setminus E_\infty$, i.e. that

$p_+(E\backslash E_\infty) < \infty$, then the two norms are still not comparable in general. We also must assume that $p_-(E\backslash E_1) > 1$. We prove this in the following theorem.

Theorem 17. *Let $E \subseteq \mathbb{R}^n$. Let $p(\cdot) \in \mathscr{P}(E)$ such that $p_-(E\backslash E_1) = 1$. Then there exists a measurable function $f : E \to R$ such that*

$$\|f\|_{p(\cdot)} = \infty \tag{9}$$

$$\||f|^{p(\cdot)-1}\|_{p'(\cdot)} < \infty \tag{10}$$

Proof. Let $E \subseteq \mathbb{R}^n$. Let $p(\cdot) \in \mathscr{P}(E)$ with $p_-(E) = 1$. by the definition of the essential infimum, there exists a sequence of sets $\{E_k\}_{k=1}^{\infty}$ with finite measure such that

1. if $x \in E_k$, then $p(x) \le \frac{1}{k} + 1$,
2. $E_k \subseteq E\backslash E_1$,
3. $E_{k+1} \subseteq E_k$ and $|E_k\backslash E_{k+1}| > 0$, and
4. $|E_k| \to 0$ as $k \to \infty$.

For each $k \in \mathbb{N}$, define $I_k = E_k\backslash E_{k+1}$. Define $f : E \to \mathbb{R}$ by for all $x \in E$,

$$f(x) = \left(\sum_{k=1}^{\infty} \frac{1}{|I_k|}\chi_{I_k}(x)\right)^{1/p(x)}.$$

To prove (9), let $\mu > 0$. Observe that as $k \to \infty$, $p_+(I_k) \to 1$. Thus there exists $K \in \mathbb{N}$ such that for all $k \ge K$, if $x \in I_k$,

$$\mu^{-p(x)} > \frac{1}{2}\mu^{-1} \tag{11}$$

Now observe that

$$
\begin{aligned}
\rho_{p(\cdot)}\left(\frac{f}{\mu}\right) &= \int_{E\setminus E_\infty} \mu^{-p(x)} \sum_{k=1}^{\infty} \frac{1}{|I_k|}\chi_{I_k}(x)dx && (p_+(I_k) < \infty \text{ for all } k)\\
&= \sum_{k=1}^{\infty} \frac{1}{|I_k|}\int_{I_k} \mu^{-p(x)}dx \\
&\geq \sum_{k=K}^{\infty} \frac{1}{|I_k|}\int_{I_k} \mu^{-p(x)}dx && \text{(remove terms)}\\
&\geq \sum_{k=K}^{\infty} \frac{1}{|I_k|}\int_{I_k} \frac{1}{2}\mu^{-1}dx && \text{(see (11))}\\
&= \sum_{k=K}^{\infty} \frac{1}{2}\mu^{-1} && (\int_{I_k} dx = |I_k|)\\
&= \infty
\end{aligned}
$$

Since $\rho_{p(\cdot)}\left(\frac{f}{\mu}\right) = \infty$ for all $\mu > 0$, then $\|f\|_{p(\cdot)} = \infty$. We now show that $\||f|^{p(\cdot)-1}\|_{p'(\cdot)}$ is finite. First note that $\{x \in E : p'(x) = \infty\} = \{x \in E : p(x) = 1\} = E_1$. This is reflected in the domain of the integral in the modular $\rho_{p'(\cdot)}$. Now consider $\mu = 2$. By our construction of the sets $\{E_k\}_{k=1}^{\infty}$, if $x \in I_k$, then $\frac{1}{p(x)-1} \geq k$, and so $-p'(x) = \frac{-p(x)}{p(x)-1} \leq -kp(x) < -k$. Thus $2^{-p'(x)} \leq 2^{-kp(x)} \leq 2^{-k}$ for all $k \in I_k$.

$$
\begin{aligned}
\rho_{p'(\cdot)}\left(\frac{|f|^{p(\cdot)-1}}{2}\right) &= \int_{E\setminus E_1} 2^{-p'(x)} \left(\sum_{k=1}^{\infty} \frac{1}{|I_k|}\chi_{I_k}(x)\right)^{p'(x)\frac{p(x)-1}{p(x)}} dx && (f(x) = 0 \text{ on } E_1)\\
&= \int_{E\setminus E_1} 2^{-p'(x)} \sum_{k=1}^{\infty} \frac{1}{|I_k|}\chi_{I_k}(x)dx && (\frac{p(x)-1}{p(x)} = \frac{1}{p'(x)})\\
&= \sum_{k=1}^{\infty} \frac{1}{|I_k|}\int_{I_k} 2^{-p'(x)}dx \\
&< \sum_{k=1}^{\infty} \frac{1}{|I_k|}\int_{I_k} 2^{-k}dx
\end{aligned}
$$

Since 2^{-k} is constant with respect to x, this last expression simplifies to a geometric series equal to 1. Thus $\rho_{p'(\cdot)}\left(\frac{|f|^{p(\cdot)-1}}{2}\right) \leq 1$ and so $\||f|^{p(\cdot)-1}\|_{p'(\cdot)} \leq 2$ which is finite. $\square$

Having illustrated that if $p_+(E\backslash E_\infty) = \infty$ or $p_-(E\backslash E_1) = 1$, then the norms $\|f\|_{p(\cdot)}$ and $\||f|^{p(\cdot)-1}\|_{p'(\cdot)}$ are not comparable in general, we turn to proving that these two norms are comparable when $1 < p_- \leq p_+ < \infty$.

Theorem 18. *Let $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \leq p_+ < \infty$, and f be measurable on E. Then $\|f\|_{p(\cdot)}$ is finite if and only if $\||f|^{p(\cdot)-1}\|_{p'(\cdot)}$ is finite. In particular*

1. $0 < \||f|^{p(\cdot)-1}\|_{p'(\cdot)} < 1$ *if and only if* $0 < \|f\|_{p(\cdot)} < 1$. *Moreover, if* $0 < \|f\|_{p(\cdot)} < 1$, *then*

$$\|f\|_{p(\cdot)}^{p_+-1} \leq \||f|^{p(\cdot)-1}\|_{p'(\cdot)} \leq \|f\|_{p(\cdot)}^{p_--1} \tag{12}$$

2. $1 \leq \||f|^{p(\cdot)-1}\|_{p'(\cdot)} < \infty$ *if and only if* $1 \leq \|f\|_{p(\cdot)} < \infty$. *Moreover, if* $1 \leq \|f\|_{p(\cdot)} < \infty$, *then*

$$\|f\|_{p(\cdot)}^{p_--1} \leq \||f|^{p(\cdot)-1}\|_{p'(\cdot)} \leq \|f\|_{p(\cdot)}^{p_+-1} \tag{13}$$

Before proving this result, we state some observations about conjugate exponent functions and the modular. Let $E \subseteq \mathbb{R}^n$ and let $p(\cdot)$ and $p'(\cdot)$ be conjugate exponent functions on E, i.e. $\dfrac{1}{p(\cdot)} + \dfrac{1}{p'(\cdot)} = 1$. Then we also have that

$$p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1} \tag{14}$$

Furthermore, if $1 < p_- \le p_+ < \infty$, then for a.e. $x \in E$ the following statements hold:

$$0 < \frac{1}{p_+ - 1} \le \frac{1}{p(x) - 1} \le \frac{1}{p_- - 1} < \infty \tag{15}$$

$$0 < p_- - 1 \le \frac{1}{p'(x) - 1} \le p_+ - 1 < \infty \tag{16}$$

Proposition 19. *Let $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \le p_+ < \infty$, and f be measurable on E. Let $\mu_0 = \||f|^{p(\cdot)-1}\|_{p'(\cdot)}$ and $\mu_p = \|f\|_{p(\cdot)}$. Then we have the following:*

$$\text{if } \mu_0 < \infty \text{ then } \int_E \left(\frac{|f(x)|}{\mu_0^{1/(p(x)-1)}} \right)^{p(x)} dx \le 1 \tag{17}$$

$$\text{if } \mu_p < \infty \text{ then } \int_E \left(\frac{|f|^{p(x)-1}}{\mu_p^{1/(p'(x)-1)}} \right)^{p'(x)} dx \le 1 \tag{18}$$

Proof. Let $0 < \mu_0 < \infty$. Observe that

$$\rho_{p'(\cdot)}\left(\frac{|f|^{p(\cdot)-1}}{\mu_0} \right) = \int_E \frac{|f(x)|^{(p(x)-1)p'(x)}}{\mu_0^{p'(x)}} dx = \int_E \frac{|f(x)|^{p(x)}}{\mu_0^{p(x)/(p(x)-1)}} dx = \int_E \left(\frac{|f(x)|}{\mu_0^{1/(p(x)-1)}} \right)^{p(x)} dx$$

By definition of $\||f|^{p(\cdot)-1}\|_{p'(\cdot)}$, $\rho_{p'(\cdot)}\left(\frac{|f|^{p(\cdot)-1}}{\mu_0} \right) \le 1$. Thus (17) holds. Now let $0 < \mu_p < \infty$. Observe that

$$\rho_{p(\cdot)}\left(\frac{|f|}{\mu_p} \right) = \int_E \left(\frac{|f|}{\mu_p} \right)^{p'(x)/(p'(x)-1)} dx = \int_E \left(\frac{|f|^{1/(p'(x)-1)}}{\mu_p^{1/(p'(x)-1)}} \right)^{p'(x)} dx = \int_E \left(\frac{|f|^{p(x)-1}}{\mu_p^{1/(p'(x)-1)}} \right)^{p'(x)} dx$$

By definition of $\|f\|_{p(\cdot)}$, $\rho_{p(\cdot)}\left(\frac{|f|}{\mu_p} \right) \le 1$. Thus (18) holds.

$\square$

With these observations, we now prove theorem 18.

Proof. (Theorem 18) We first prove case 1, i.e. that $0 < \||f|^{p(\cdot)-1}\|_{p'(\cdot)} < 1$ if and only if $0 < \|f\|_{p(\cdot)} < 1$. Assume $0 < \||f|^{p(\cdot)-1}\|_{p'(\cdot)} < 1$. Let $\mu_0 = \||f|^{p(\cdot)-1}\|_{p'(\cdot)}$. By (15), $\mu_0^{1/(p_+-1)} \geq \mu_0^{1/(p(x)-1)}$, and so we have

$$\int_E \left(\frac{|f(x)|}{\mu_0^{1/(p_+-1)}} \right)^{p(x)} dx \leq \int_E \left(\frac{|f(x)|}{\mu_0^{1/(p(x)-1)}} \right)^{p(x)} dx$$

By (17), the right side of this last inequality is at most 1. Thus, $\|f\|_{p(\cdot)} \leq \mu_0^{1/(p_+-1)}$, or equivalently, $\|f\|_{p(\cdot)}^{p_+-1} \leq \||f|^{p(\cdot)-1}\|_{p'(\cdot)}$. Therefore, $0 < \|f\|_{p(\cdot)} < 1$. The proof of the converse is similar. Assume $0 < \|f\|_{p(\cdot)} < 1$. Let $\mu_p = \|f\|_{p(\cdot)}$. Now note that since $0 < \mu_p < 1$, (16) implies that $\mu_p^{p_--1} \geq \mu_p^{1/(p'(x)-1)}$. Thus, by monotonicity, we have

$$\int_E \left(\frac{|f|^{p(x)-1}}{\mu_p^{p_--1}} \right)^{p'(x)} dx \leq \int_E \left(\frac{|f|^{p(x)-1}}{\mu_p^{1/(p'(x)-1)}} \right)^{p'(x)} dx$$

By (18), the right side of this last inequality is at most 1. Thus $\||f|^{p(\cdot)-1}\|_{p'(\cdot)} \leq \mu_p^{p_--1}$ and so $0 < \||f|^{p(\cdot)-1}\|_{p'(\cdot)} < 1$. Moreover, we have

$$\|f\|_{p(\cdot)}^{p_+-1} \leq \||f|^{p(\cdot)-1}\|_{p'(\cdot)} \leq \|f\|_{p(\cdot)}^{p_--1}$$

Hence, (12) holds. For the second case, suppose that $1 \leq \||f|^{p(\cdot)-1}\|_{p'(\cdot)} < \infty$. Let $\mu_0 = \||f|^{p(\cdot)-1}\|_{p'(\cdot)}$. Since $\mu_0 \geq 1$, (15) implies $\mu_0^{1/(p_--1)} \geq \mu_0^{1/(p(x)-1)}$, which gives

$$\int_E \left(\frac{|f|}{\mu_0^{1/(p_--1)}} \right)^{p(x)} dx \leq \int_E \left(\frac{|f|}{\mu_0^{1/(p(x)-1)}} \right)^{p(x)} dx$$

By (17), the right side of the last inequality is at most 1, we have that $\|f\|_{p(\cdot)} \leq \mu_0^{1/(p_- -1)}$, or equivalently $\|f\|_{p(\cdot)}^{p_- -1} \leq \||f|^{p(\cdot)-1}\|_{p'(\cdot)}$. Thus, $\|f\|_{p(\cdot)}$ is finite. If $\|f\|_{p(\cdot)} < 1$, then from case 1, we would have $\mu_0 < 1$. This would contradict the assumption that $\mu_0 \geq 1$, Thus, $\|f\|_{p(\cdot)} \geq 1$.

To prove the converse, assume $1 \leq \|f\|_{p(\cdot)} < \infty$. Let $\mu_p = \|f\|_{p(\cdot)}$. Since $\mu_p \geq 1$, (16) implies $\mu_p^{p_+ -1} \geq \mu_p^{1/(p'(x)-1)}$. Thus, we have

$$\int_E \left(\frac{|f|^{p(x)-1}}{\lambda_p^{p_+ -1}} \right)^{p'(x)} dx \leq \int_E \left(\frac{|f|^{p(x)-1}}{\lambda_p^{1/(p'(x)-1)}} \right)^{p'(x)} dx$$

By (18), the right side of the last inequality is at most 1, we have that $\||f|^{p(\cdot)-1}\|_{p'(\cdot)} \leq \lambda_p^{p_+ -1}$, or equivalently $\||f|^{p(\cdot)-1}\|_{p'(\cdot)} \leq \|f\|_{p(\cdot)}^{p_+ -1}$. Thus $\||f|^{p(\cdot)-1}\|_{p'(\cdot)}$ is finite. If $\||f|^{p(\cdot)-1}\|_{p'(\cdot)} < 1$, then from case 1, we would have $\mu_p < 1$. This would contradict the assumption that $\mu_p \geq 1$. Thus $\||f|^{p(\cdot)-1}\|_{p'(\cdot)} \geq 1$. Moreover, we have

$$\|f\|_{p(\cdot)}^{p_- -1} \leq \||f|^{p(\cdot)-1}\|_{p'(\cdot)} \leq \|f\|_{p(\cdot)}^{p_+ -1}$$

Hence, (13) holds. $\square$

While this result requires more assumptions than in the constant exponent setting, this doesn't necessarily mean that theorem 1 cannot be translated into the variable exponent setting. One of the necessary assumptions in theorem 1 is that $1 < p < \infty$. This parallels the assumption in theorem 18 that $1 < p_- \leq p_+ < \infty$.

3.2 Completeness

As discussed in section 2.2, the completeness of $H_Q^{1,p}(v;E)$ rested on the completeness of $\mathcal{L}_Q^p(E)$. To translate the results of [5] into the variable setting, we need to show $\mathcal{L}_Q^{p(\cdot)}(E)$ is complete. In order to do this, we first define $\mathcal{L}_Q^{p(\cdot)}(E)$ and introduce some important facts.

Definition 20. Let $E \subseteq \mathbb{R}^n$, $p(\cdot) \in \mathscr{P}(E)$, and $Q : E \to \mathcal{S}_n$ be a measurable $n \times n$ matrix valued function. The form-weighted vector-valued variable Lebesgue space $\mathcal{L}_Q^{p(\cdot)}(E)$ is the collection of all measurable $\mathbb{R}^n$ valued functions $\mathbf{f} = (f_1, \cdots, f_n) : E \to \mathbb{R}^n$ satisfying

$$||\mathbf{f}||_{\mathcal{L}_Q^{p(\cdot)}(E)} = \left\| |\sqrt{Q(x)}\mathbf{f}(x)| \right\|_{p(\cdot)} < \infty.$$

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $1 \leq r \leq \infty$, define the ℓ^r norm on $\mathbb{R}^n$ by $|x|_r = \left(\sum_{j=1}^n |x_j|^r\right)^{1/r}$ for $r < \infty$ and $|x|_\infty = \sup_{1 \leq j \leq n} |x_j|$. As mentioned in section 2.1, when $r = 2$, we have the Euclidean norm, and denote it by $|\cdot|_2 = |\cdot|$. Recall that in finite dimensions, all norms are equivalent. In particular, the ℓ^1, ℓ^2, and ℓ^∞ norms have the following equivalences on $\mathbb{R}^n$.

Lemma 21. *Let $x \in \mathbb{R}^n$. Then the following equivalences hold:*

$$|x|_2 \leq |x|_1 \leq \sqrt{n}|x|_2,$$

$$|x|_\infty \leq |x|_2 \leq \sqrt{n}|x|_\infty$$

$$|x|_\infty \leq |x|_1 \leq n|x|_\infty$$

Recall that every finite, self-adjoint matrix is diagonalizable. We extend this to matrix-valued functions.

Lemma 22 (Lemma 2.3.5, [11])**.** *Let Q be an finite, self-adjoint matrix whose entries are Lebesgue measurable functions on some domain E. Then for every $x \in E$, $Q(x)$ is diagonalizable, i.e. there exists a matrix U whose entries are Lebesgue measurable functions on E such that $U^T QU$ is a diagonal matrix and $U(x)$ is unitary for every $x \in E$. Equivalently, there exists a diagonal matrix function $D(x)$ such that for almost every $x \in E$*

$$Q(x) = U^T(x)D(x)U(x).$$

In particular, given such a matrix Q, we can define its square root by

$$\sqrt{Q(x)} = U^T(x)\sqrt{D(x)}U(x),$$

where $\sqrt{D(x)}$ takes the square root of every entry along the diagonal.

Remark 23. Given a measurable matrix function, $Q : E \to \mathcal{S}_n$, let $\{\lambda_j(x)\}_{j=1}^n$ be the eigenvalues of $Q(x)$ and $\{\mathbf{v}_j(x)\}_{j=1}^n$ be corresponding orthonormal eigenvectors. The eigenvalues, $\lambda_j(x)$ are measurable since $Q(x)$ is measuarable, while the eigenvectors, $\mathbf{v}_j(x)$ may be chosen to be Lebesgue measurable. The proof of this is given in remark 5 of [13].

We need one more lemma before proving $\mathcal{L}_Q^{p(\cdot)}(E)$ is complete. In section 2.3, we stated that $L^{p(\cdot)}(E)$ is a Banach space. In order to show that $\mathcal{L}_Q^{p(\cdot)}(E)$ is complete, we must use the fact that $L^{p(\cdot)}(v;E)$ is a Banach space.

Lemma 24. *Let $E \subseteq \mathbb{R}^n$ and v be a weight on E. Let $p(\cdot) \in \mathscr{P}(E)$. Then $L^{p(\cdot)}(v;E)$ is a Banach space.*

Proof. Let $E_0 \subseteq E$ be the support of the weight v. Then $L^{p(\cdot)}(v;E) = L^{p(\cdot)}(v;E_0)$. Define the mapping $I : L^{p(\cdot)}(v;E_0) \to L^{p(\cdot)}(E_0)$ by $I(f) = fv$. This mapping is linear and is invertible, with inverse $I^{-1}(g) = \dfrac{g}{v}$. Note that since E_0 is the support of v, $\frac{g}{v}$ is defined. Moreover, $\|f\|_{L^{p(\cdot)}(v;E_0)} = \|fv\|_{L^{p(\cdot)}(E_0)}$, and so the map I is an isometry. Thus I is an isometric isomorphism from $L^{p(\cdot)}(v;E_0)$ to $L^{p(\cdot)}(E_0)$. Since $L^{p(\cdot)}(v;E) = L^{p(\cdot)}(v;E_0)$, we also have that $L^{p(\cdot)}(v;E)$ and $L^{p(\cdot)}(E_0)$ are isometrically isomorphic. Therefore, since $L^{p(\cdot)}(E_0)$ is a Banach space, so is $L^{p(\cdot)}(v;E)$. □

With these lemmas, we can prove that $\mathcal{L}^{p(\cdot)}_Q(E)$ is complete.

Theorem 25. *Given $E \subseteq \mathbb{R}^n$, $p(\cdot) \in \mathscr{P}(E)$ and a measurable $n \times n$ matrix function $Q : E \to \mathcal{S}^n$, then $\mathcal{L}^{p(\cdot)}_Q(E)$ is a Banach space.*

Proof. Since $Q(x)$ is finite and self-adjoint, by lemma 22, $Q(x)$ is diagonalizable. By remark 23, the eigenvalues of $Q(x)$ are measurable and the corresponding unit eigenvectors may be chosen to be measurable. Denote by $\lambda_1(x), \dots, \lambda_n(x)$ the measurable eigenvalues of $Q(x)$ and choose measurable eigenvectors $\mathbf{v}_1(x), \dots, \mathbf{v}_n(x)$ with $|\mathbf{v}_j(x)| = 1$ for almost every $x \in E$ and for $1 \le j \le n$. Hence, $\{\mathbf{v}_j(x)\}_{j=1}^n$ forms a basis for $\mathbb{R}^n$ for almost every $x \in E$. Fix $\mathbf{f} \in \mathcal{L}^{p(\cdot)}_Q(E)$. We now write $\mathbf{f}$ as

$$\mathbf{f}(x) = \sum_{j=1}^{n} \tilde{f}_j(x)\mathbf{v}_j(x)$$

where $\tilde{f}_j = \mathbf{f}^T \mathbf{v}_j$ is the jth component of $\mathbf{f}$ with respect to the basis $\{\mathbf{v}_j\}_{j=1}^n$. Completeness follows once we establish the equivalence of norms: for all $\mathbf{f} \in \mathcal{L}^{p(\cdot)}_Q(E)$,

$$\frac{1}{n}\sum_{j=1}^{n} \|\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)} \le \|\mathbf{f}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} \le \sum_{j=1}^{n} \|\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)} \tag{19}$$

Indeed, suppose inequality (19) holds. Let $\{\mathbf{f}_k\}_{k=1}^\infty$ be a Cauchy sequence in $\mathcal{L}^{p(\cdot)}_Q(E)$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for every $l, m > N$, $\|\mathbf{f}_l - \mathbf{f}_m\|_{\mathcal{L}^{p(\cdot)}_Q(E)} < \varepsilon/n$. Let $l, m > N$. Then by inequality (19),

$$\sum_{j=1}^n \|(\mathbf{f}_l - \mathbf{f}_m)^T \mathbf{v}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)} \leq n\|\mathbf{f}_l - \mathbf{f}_m\|_{\mathcal{L}^{p(\cdot)}_Q(E)} < \varepsilon.$$

Thus for each $j = 1, \ldots, n$, $\{\mathbf{f}_k^T \mathbf{v}_j\}_{k=1}^\infty$ is Cauchy in $L^{p(\cdot)}(\lambda_j^{1/2};E)$. By lemma 24, $L^{p(\cdot)}(\lambda_j^{1/2};E)$ is complete, and so there exists $\tilde{g}_j \in L^{p(\cdot)}(\lambda_j^{1/2};E)$ such that as $k \to \infty$,

$$\|\mathbf{f}_k^T \mathbf{v}_j - \tilde{g}_j\|_{L^{p(\cdot)}(\lambda^{1/2};E)} \to 0.$$

Define $\mathbf{g} : E \to \mathbb{R}^n$ by for $x \in E$, $\mathbf{g}(x) = \sum_{j=1}^n \tilde{g}_j(x)\mathbf{v}_j(x)$. Now by (19), we have

$$\|\mathbf{f}_k - \mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} \leq \sum_{j=1}^n \|\mathbf{f}_k^T \mathbf{v}_j - \tilde{g}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)}.$$

Since for each $j = 1, \ldots, n$ we have $\|\mathbf{f}_k^T \mathbf{v}_j - \tilde{g}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)} \to 0$ as $k \to \infty$, we have that $\mathbf{f}_k \to \mathbf{g}$ in $\mathcal{L}^{p(\cdot)}_Q(E)$ in norm. Since for each $j = 1, \ldots, n$, we have $\tilde{g}_j \in L^{p(\cdot)}(\lambda_j^{1/2};E)$, then by (19), $\mathbf{g} \in \mathcal{L}^{p(\cdot)}_Q(E)$. Therefore, $\mathcal{L}^{p(\cdot)}_Q(E)$ is complete.

It remains to prove inequality (19). We first establish a pointwise equality: for almost every $x \in E$,

$$|\sqrt{Q(x)}\mathbf{f}(x)| = \left(\sum_{j=1}^n |\tilde{f}_j(x)|^2 \lambda_j(x)\right)^{1/2}. \tag{20}$$

Observe that for almost every $x \in E$,

$$\begin{aligned}
|\sqrt{Q(x)}\mathbf{f}(x)|^2 &= \sum_{j=1}^{n} |\tilde{f}_j(x)\sqrt{Q(x)}\mathbf{v}_j(x)|^2 && \text{(Pythagorean theorem on } \mathbb{R}^n) \\
&= \sum_{j=1}^{n} |\tilde{f}_j(x)|^2 \mathbf{v}_j^T(x)\sqrt{Q^T(x)}\sqrt{Q(x)}\mathbf{v}_j(x) && (\forall \mathbf{y} \in \mathbb{R}^n, |\mathbf{y}|^2 = \mathbf{y}^T\mathbf{y}) \\
&= \sum_{j=1}^{n} |\tilde{f}_j(x)|^2 \mathbf{v}_j^T(x)Q(x)\mathbf{v}_j(x) && (Q \text{ self-adjoint}) \\
&= \sum_{j=1}^{n} |\tilde{f}_j(x)|^2 \lambda_j(x)\mathbf{v}_j^T(x)\mathbf{v}_j(x) && (Q(x)\mathbf{v}_j(x) = \lambda_j(x)\mathbf{v}_j(x)) \\
&= \sum_{j=1}^{n} |\tilde{f}_j(x)|^2 \lambda_j(x) && (|\mathbf{v}_j(x)| = 1)
\end{aligned}$$

Taking the square root yields the desired equation. We now show that the $\mathcal{L}_Q^{p(\cdot)}$ norm is equivalent to a sum of weighted $L^{p(\cdot)}$ norms. Define $\tilde{F} : E \to \mathbb{R}^n$ by for $x \in E$, $\tilde{F}(x) = (|\tilde{f}_1(x)|\lambda_1^{1/2}(x), \ldots, |\tilde{f}_n(x)|\lambda_n^{1/2}(x))$. Applying the pointwise equality (20) and our definition of $\tilde{F}$ we have that

$$\|\mathbf{f}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} = \left\| |\sqrt{Q(x)}\mathbf{f}(x)| \right\|_{p(\cdot)} = \left\| |\tilde{F}(x)| \right\|_{p(\cdot)}.$$

We now prove both inequalities in inequality (19). To show $\|\mathbf{f}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \leq \sum_{j=1}^n \|\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)}$, we apply lemma 21 and the triangle inequality to get

$$\left\| |\tilde{F}(x)| \right\|_{p(\cdot)} \leq \left\| |\tilde{F}(x)|_1 \right\|_{p(\cdot)} \leq \sum_{j=1}^{n} \|\tilde{f}_j \lambda_j^{1/2}\|_{p(\cdot)} = \sum_{j=1}^{n} \|\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)}.$$

To show the reverse inequality, we apply lemma 21 to get $\| |\tilde{F}(x)| \|_{p(\cdot)} \geq \| |\tilde{F}(x)|_\infty \|_{p(\cdot)}$. Now we use the fact that $\sum_{j=1}^n 1 = n$ and the definition of $|\cdot|_\infty$ to get

$$\left\| |\tilde{F}(x)|_\infty \right\|_{p(\cdot)} = \frac{1}{n}\sum_{j=1}^{n} \left\| |\tilde{F}(x)|_\infty \right\|_{p(\cdot)} \geq \frac{1}{n}\sum_{j=1}^{\infty} \| |\tilde{f}_j(x)\lambda_j^{1/2}(x)| \|_{p(\cdot)} = \frac{1}{n}\sum_{j=1}^{n} \|\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)}.$$

Thus we have proven inequality (19). $\square$

3.3 Separable and Reflexive

In order to use Minty's theorem from section 2.2, we must show that the variable version of $H_Q^{1,p(\cdot)}(v;E)$ is reflexive and separable. This is proven in chapter 4. However, the proof relies on the fact that $\mathcal{L}_Q^{p(\cdot)}(E)$ is reflexive and separable. We now turn our attention to proving $\mathcal{L}_Q^{p(\cdot)}(E)$ is separable and reflexive. To do so, we will utilize the norm equivalence (19) established in theorem 25. We will also rely heavily on the fact that $L^{p(\cdot)}(E)$ is separable if E is open and $p_+ < \infty$, as well as that $L^{p(\cdot)}(E)$ is reflexive if $1 < p_- \leq p_+ < \infty$. We begin by exploring the separable property. We state that $L^{p(\cdot)}(E)$ is separable, then show that the weighted version is also separable.

Theorem 26. *[4, Theorem 2.78] Given an open set $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$, then $L^{p(\cdot)}(E)$ is separable if and only if $p_+ < \infty$.*

Theorem 27. *Given a open set $E \subseteq \mathbb{R}^n$, a weight v on E, and $p(\cdot) \in \mathscr{P}(E)$, then $L^{p(\cdot)}(v;E)$ is separable if $p_+ \infty$.*

Proof. By theorem 26, $L^{p(\cdot)}(E)$ is separable, and so there is a countable, dense subset $D \subseteq L^{p(\cdot)}(E)$. Let $\varepsilon > 0$ and $f \in L^{p(\cdot)}(v;E)$. Then $fv \in L^{p(\cdot)}(E)$, and so there is an element $d \in D$ such that $\|fv - d\|_{p(\cdot)} < \varepsilon$. Let $\mu = \|fv - d\|_{p(\cdot)}$. Let E_0 be the support of v. Define

$d_0 : E \to \mathbb{R}$ by

$$d_0(x) = \begin{cases} \frac{d}{v}(x) & \text{if } \in E_0 \\ \\ 0 & \text{if } x \notin E_0 \end{cases}.$$

Observe that

$$\begin{aligned} \rho\left(\frac{fv - d_0 v}{\mu}\right) &= \int_{E_0} \left|\frac{fv - d_0 v}{\mu}\right|^{p(x)} dx && (v(x) = 0 \text{ for } x \notin E_0) \\ &= \int_{E_0} \left|\frac{fv - d}{\mu}\right|^{p(x)} dx && (d_0 v(x) = d(x) \text{ in } E_0) \\ &\leq \int_{E} \left|\frac{fv - d}{\mu}\right|^{p(x)} dx && (E_0 \subseteq E) \\ &\leq 1 && (\mu = \|fv - d\|_{p(\cdot)}). \end{aligned}$$

Thus $\|fv - d_0 v\|_{p(\cdot)} \leq \mu < \varepsilon$. Thus $\|f - d_0\|_{L^{p(\cdot)}(v;E)} < \varepsilon$. Hence, the collection of all possible d_0 defined as before is dense in $L^{p(\cdot)}(v;E)$. Since D is countable, so is this collection. Thus $L^{p(\cdot)}(v;E)$ is separable if E is open and $p_+ < \infty$. $\square$

We now prove that $\mathcal{L}^{p(\cdot)}_Q(E)$ is separable by leveraging the previous theorem and the norm equivalence (19).

Theorem 28. *Given an open set $E \subseteq \mathbb{R}^n$, $p(\cdot) \in \mathscr{P}(E)$, and a measurable matrix function $Q : E \to \mathcal{S}_n$, if $p_+ < \infty$, then $\mathcal{L}^{p(\cdot)}_Q(E)$ is separable.*

Proof. As in the proof of theorem 25, we denote the measurable eigenvalues of $Q(x)$ by $\lambda_1(x), \ldots, \lambda_n(x)$ and choose corresponding measurable eigenvectors $\mathbf{v}_1(x), \ldots, \mathbf{v}_n(x)$ with $|\mathbf{v}_j(x)| = 1$ for almost every $x \in E$ and for $1 \leq j \leq n$. Then $\{\mathbf{v}_j(x)\}_{j=1}^n$ forms an orthonormal basis for $\mathbb{R}^n$ for almost every $x \in E$. Let $\mathbf{f} \in \mathcal{L}^{p(\cdot)}_Q(E)$. We write $\mathbf{f}$ as

$$\mathbf{f}(x) = \sum_{j=1}^{n} \tilde{f}_j(x)\mathbf{v}_j(x)$$

where $\tilde{f}_j = \mathbf{f}^T \mathbf{v}_j$ is the jth component of $\mathbf{f}$ with respect to the basis $\{\mathbf{v}_j\}_{j=1}^n$.

Let $\varepsilon > 0$. From lemma 27, $L^{p(\cdot)}(\lambda_j^{1/2};E)$ is separable, and so for each j there is a countable, dense subset $D_j \subseteq L^{p(\cdot)}(\lambda_j^{1/2};E)$. Thus, for each $j = 1,\dots,n$, there exists $d_j \in D_j$ such that

$$\|\tilde{f}_j - d_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)} < \varepsilon/n.$$

Define $\mathbf{d} \in \bigtimes_{j=1}^n D_j$ by $\mathbf{d} = \sum_{j=1}^n d_j \mathbf{v}_j$. Then by the norm equivalence (19),

$$\|\mathbf{f} - \mathbf{d}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} \leq \sum_{j=1}^{n} \|\tilde{f}_j - d_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)} < \varepsilon.$$

Thus, $\bigtimes_{j=1}^n D_j$ is a dense subset of $\mathcal{L}^{p(\cdot)}_Q(E)$. Since the finite cartesian product of countable sets is countable, we have that $\mathcal{L}^{p(\cdot)}_Q(E)$ is separable. □

We now turn to proving $\mathcal{L}^{p(\cdot)}_Q(E)$ is reflexive. We first define the reflexive property and present some useful characterizations of reflexivity. We then present two theorems that will simplify our proof.

Definition 29. Let X be a Banach space. The dual space by X^* is the set of all bounded, linear functionals on X. Define the canonical linear isometry $\mathcal{F} : X \to X^{**}$ by

$$\langle \mathcal{F}(x), f \rangle = f(x).$$

If $\mathcal{F}$ is surjective, then X is reflexive.

Closed subspaces and product spaces of reflexive Banach spaces are also reflexive. Moreover, given two Banach spaces X and Y, if $T : X \to Y$ is an isomorphism, then X is reflexive if and only if Y is reflexive. In this context, T is an isomorphism provided T is bijective, linear, and T and T^{-1} are continuous. We now state when variable Lebesgue spaces are reflexive.

Theorem 30. *[4, Corollary 2.81] Given $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$, $L^{p(\cdot)}(E)$ is reflexive if and only if $1 < p_- \leq p_+ < \infty$.*

Theorem 31. *Given $E \subseteq \mathbb{R}^n$, a weight v on E, and $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \leq p_+ < \infty$, $L^{p(\cdot)}(v;E)$ is reflexive if $1 < p_- \leq p_+ < \infty$.*

The proof of theorem 31 is identical to the proof of completeness in lemma 24 found in section 3.2. This is because a closed subspace of a reflexive Banach space is reflexive. We now to prove that $\mathcal{L}^{p(\cdot)}_Q(E)$ is reflexive by showing it is isomorphic to a product of reflexive Banach spaces.

Theorem 32. *Given a set $E \subseteq \mathbb{R}^n$, $p(\cdot) \in \mathscr{P}(E)$, and a measurable matrix function $Q : E \to \mathcal{S}_n$, if $1 < p_- \leq p_+ < \infty$, then $\mathcal{L}^{p(\cdot)}_Q(E)$ is reflexive.*

Proof. Let $\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)$ be the eigenvalues of $Q(x)$, and let $\mathbf{v}_1(x), \mathbf{v}_2(x), \ldots, \mathbf{v}_n(x)$ be the correspoding unit eigenvectors. Let $\mathbf{f} \in \mathcal{L}^{p(\cdot)}_Q(E)$. Then we can write $\mathbf{f}$ as

$$\mathbf{f}(x) = \sum_{j=1}^{n} \tilde{f}_j(x)\mathbf{v}_j(x),$$

where $\tilde{f}_j = \mathbf{f}^T \mathbf{v}_j$. This induces a map $T : \mathcal{L}^{p(\cdot)}_Q(E) \to \bigtimes_{j=1}^{n} L^{p(\cdot)}(\lambda_j^{1/2}; E)$ defined by $T(\mathbf{f}) = (\tilde{f}_1, \ldots, \tilde{f}_j)$. Observe that T is bijective because of the norm equivalence (19). That is,

$\|vf\|_{\mathcal{L}_Q^{p(\cdot)}(E)}$ is finite if and only if the product space norm $\sum_{j=1}^n \|\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)}$ is finite, and so $\mathbf{f} \in \mathcal{L}_Q^{p(\cdot)}(E)$ if and only if $T(\mathbf{f}) \in \bigtimes_{j=1}^n L^{p(\cdot)}(\lambda_j^{1/2};E)$.

To see that T is linear, observe that for all $\mathbf{f},\mathbf{g} \in \mathcal{L}_Q^{p(\cdot)}(E)$ and $\alpha \in \mathbb{R}$, $(\mathbf{f}+\alpha\mathbf{g})^T\mathbf{v}_j = \mathbf{f}^T\mathbf{v}_j + \alpha\mathbf{g}^T\mathbf{v}_j$. Thus,

$$T(\mathbf{f}+\alpha\mathbf{g}) = (\tilde{f}_1+\alpha\tilde{g}_1,\ldots,\tilde{f}_n+\alpha\tilde{g}_n) = T(\mathbf{f})+\alpha T(\mathbf{g}).$$

Thus, T is linear. We now show that T is continuous by showing if $\mathbf{f}_k \to \mathbf{f}$ in $\mathcal{L}_Q^{p(\cdot)}(E)$, then $T(\mathbf{f}_k) \to T(\mathbf{f})$ in $\bigtimes_{j=1}^n L^{p(\cdot)}(\lambda_j^{1/2};E)$. Note that by the product norm definition,

$$\|T(\mathbf{f}_k)-T(\mathbf{f})\|_{\bigtimes_{j=1}^n L^{p(\cdot)}(\lambda_j^{1/2};E)} = \sum_{j=1}^n \|\tilde{f}_{jk}-\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2});E}.$$

By the norm equivalence (19), this product norm is bounded by $n\|\mathbf{f}_k-\mathbf{f}\|_{L^{p(\cdot)}(v;E)}$. Since $\mathbf{f}_k \to \mathbf{f}$ in $\mathcal{L}_Q^{p(\cdot)}(E)$, this upper bounded tends to zero as $k \to \infty$. Hence, so does $\|T(\mathbf{f}_k)-T(\mathbf{f})\|_{\bigtimes_{j=1}^n L^{p(\cdot)}(\lambda_j^{1/2};E)}$. Thus, T is continuous. The norm equivalence (19) also shows that T^{-1} is continuous, since $\|\mathbf{f}_k-\mathbf{f}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}$ is bounded by $\sum_{j=1}^n \|\tilde{f}_{jk}-\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)}$.

Since $\mathcal{L}_Q^{p(\cdot)}(E)$ is isomorphic to the product of reflexive space, $\bigtimes_{j=1}^n L^{p(\cdot)}(\lambda_j^{1/2};E)$, and products of reflexive spaces are reflexive, then so is $\mathcal{L}_Q^{p(\cdot)}(E)$. $\square$

We have shown that for $1 < p_- \leq p_+ < \infty$, the norms $\|f\|_{p(\cdot)}$ and $\||f|^{p(\cdot)-1}\|_{p'(\cdot)}$ are comparable, as well as that $\mathcal{L}_Q^{p(\cdot)}(E)$ is complete, is separable if E is open and $p_+ < \infty$, and is reflexive for $1 < p_- \leq p_+ < \infty$. With this result, we would expect that theorem 1 will translate into the variable exponent setting. However, it does not. We devote the next chapter to exploring this.

CHAPTER 4

Exploration of the $p(\cdot)$-Neumann and $p(\cdot)$-Poincaré Properties

4.1 Definitions

We begin our exploration by building the variable version of the Sobolev space $H_Q^{1,p}(v;E)$. Recall that a weight v on a set E is a non-negative measurable functions in $L^1_{\text{loc}}(E)$ with $v(x) < \infty$ for almost every $x \in E$. As we did with the constant exponent version in section 2.2, we define $H_Q^{1,p(\cdot)}(v;E)$ to be a collection of equivalence classes of Cauchy sequences of $C^1(\overline{E})$ functions.

Definition 33. Given a bounded, open set $E \subseteq \mathbb{R}^n$, a weight v on E, a measurable matrix function $Q : E \to \mathcal{S}_n$ and $p(\cdot) \in \mathscr{P}(E)$, the Sobolev space $H_Q^{1,p(\cdot)}(v;E)$ is the abstract completion of $C^1(\overline{E})$ with respect to the norm

$$\|f\|_{H_Q^{1,p(\cdot)}(v;E)} = \|f\|_{L^{p(\cdot)}(v;E)} + \|\nabla f\|_{\mathcal{L}_Q^{p(\cdot)}(E)}$$

We can uniquely represent elements $f \in H_Q^{1,p(\cdot)}(v;E)$ by their representative pairs $(u,\mathbf{g})_f \in L^{p(\cdot)}(v;E) \times \mathcal{L}_Q^{p(\cdot)}(E)$. Since $L^{p(\cdot)}(v;E)$ and $\mathcal{L}_Q^{p(\cdot)}(E)$ are Banach Spaces, are separable if E is open and $p_+ < \infty$, and reflexive if $1 < p_- \leq p_+ < \infty$, so is $H_Q^{1,p(\cdot)}(v;E)$.

Theorem 34. *Let $E \subseteq \mathbb{R}^n$ be a bounded, open set, $p(\cdot) \in \mathscr{P}(E)$, and $Q : E \to \mathcal{S}_n$ be a measurable matrix function. Then $H_Q^{1,p(\cdot)}(v;E)$ is a Banach Space. Moreover, if $p_+ < \infty$, then it is separable, and reflexive if* $1 < p_- \leq p_+ < \infty$.

Proof. It suffices to show that $H_Q^{1,p(\cdot)}(v;E)$ is isometrically isomorphic to a closed subspace of $L^{p(\cdot)}(v;E) \times \mathcal{L}_Q^{p(\cdot)}(E)$. Define the map $I : H_Q^{1,p(\cdot)}(v;E) \to L^{p(\cdot)}(v;E) \times \mathcal{L}_Q^{p(\cdot)}(E)$ by

$$I(f) = I((u_n, \nabla u_n)) = (u, \mathbf{g})$$

where $u_n \to u$ in $L^{p(\cdot)}(v;E)$ norm and $\nabla u_n \to \mathbf{g}$ in $\mathcal{L}_Q^{p(\cdot)}(E)$ norm.

Let $f = \{(u_n, \nabla u_n)\}$ and $h = \{(w_n, \nabla w_n)\}$ be equivalence classes of Cauchy sequences of $C^1(\overline{E})$ functions, from $H_Q^{1,p(\cdot)}(v;E)$. Suppose $u_n \to u$ and $w_n \to w$ in $L^{p(\cdot)}(v;E)$ and $\nabla u_n \to \mathbf{g}$ and $\nabla w_n \to \mathbf{h}$ in $\mathcal{L}_Q^{p(\cdot)}(E)$. We will show that I is a linear isometry. Let $\alpha \in \mathbb{R}$. Observe that

$$
\begin{aligned}
I(f+\alpha h) &= I((u_n, \nabla u_n) + \alpha(w_n, \nabla w_n)) \\
&= I((u_n + \alpha w_n, \nabla(u_n + \alpha w_n))) \\
&= (u + \alpha w, \mathbf{g} + \alpha \mathbf{h}) \\
&= (u, \mathbf{g}) + \alpha(w, \mathbf{h}) \\
&= I(f) + \alpha I(h)
\end{aligned}
$$

Thus I is linear. To see that I is an isometry, observe that

$$
\begin{aligned}
\|I(f)\|_{L^{p(\cdot)}(v;E)\times\mathcal{L}_Q^{p(\cdot)}(E)} &= \|u\|_{L^{p(\cdot)}(v;E)} + \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \\
&= \lim_{n\to\infty}(\|u_n\|_{L^{p(\cdot)}(v;E)} + \|\nabla u_n\|_{\mathcal{L}_Q^{p(\cdot)}(E)}) \\
&= \lim_{n\to\infty}\|f\|_{H_Q^{1,p(\cdot)}(v;E)} \\
&= \|f\|_{H_Q^{1,p(\cdot)}(v;E)}
\end{aligned}
$$

Hence, I is a linear isometry and so its image is a closed subspace of $L^{p(\cdot)}(v;E) \times \mathcal{L}_Q^{p(\cdot)}(E)$. Consequently, $H_Q^{1,p(\cdot)}(v;E)$ is isometrically isomorphic to a closed subspace of $L^{p(\cdot)}(v;E) \times \mathcal{L}_Q^{p(\cdot)}(E)$, and thus every element of $H_Q^{1,p(\cdot)}(v;E)$ can be uniquely represented by an element of $L^{p(\cdot)}(v;E) \times \mathcal{L}_Q^{p(\cdot)}(E)$. Since $\mathcal{L}_Q^{p(\cdot)}(E) \times L^{p(\cdot)}(v;E)$ is a Banach space, so is $H_Q^{1,p(\cdot)}(v;E)$. Likewise, $H_Q^{1,p(\cdot)}(v;E)$ is reflexive if $1 < p_- \leq p_+ < \infty$, and since E is open, $H_Q^{1,p(\cdot)}(v;E)$ is separable if $p_+ < \infty$. □

In section 2.2, we used the mean-zero subspace of $H_Q^{1,p}(v;E)$ to define weak solutions. We do the same in the variable exponent setting. The mean-zero subspace of $H_Q^{1,p(\cdot)}(v;E)$ is defined by

$$\tilde{H}_Q^{1,p(\cdot)}(v;E) = \{(u,\mathbf{g}) \in H_Q^{1,p(\cdot)}(v;E) : \int_E u(x)v(x)dx = 0\}.$$

In the constant exponent setting, the mean-zero subspace of $H_Q^{1,p}(v;E)$ inherits the properties needed to apply Minty's theorem. This also happens in the variable exponent setting.

Theorem 35. *Given a bounded, open set $E \subseteq \mathbb{R}^n$, $p(\cdot) \in \mathscr{P}(E)$, and a measurable matrix function $Q : E \to \mathcal{S}_n$, then $\tilde{H}_Q^{1,p(\cdot)}(v;E)$ is a Banach space. Furthermore, $\tilde{H}_Q^{1,p(\cdot)}(v;E)$ is separable if $p_+ < \infty$ and reflexive if $1 < p_- \leq p_+ < \infty$.*

Proof. Since $H_Q^{1,p(\cdot)}(v;E)$ is a normed linear space, it is also a metric space. Every subspace of a separable metric space is separable, and so $\tilde{H}_Q^{1,p(\cdot)}(v;E)$ is immediately separable if $p_+ < \infty$.

To show that $\tilde{H}_Q^{1,p(\cdot)}(v;E)$ is a Banach space, it suffices to show that $\tilde{H}_Q^{1,p(\cdot)}(v;E)$ is a closed subspace of the Banach space $H_Q^{1,p(\cdot)}(v;E)$. Let $p(\cdot) \in \mathscr{P}(E)$. We now show that $\tilde{H}_Q^{1,p(\cdot)}(v;E)$ is closed. Let $\{(u_j,\mathbf{g}_j)\}_{j=1}^{\infty}$ be a Cauchy sequence in $\tilde{H}_Q^{1,p(\cdot)}(v;E)$. Since $H_Q^{1,p(\cdot)}(v;E)$ is complete, there is an element $(u,\mathbf{g}) \in H_Q^{1,p(\cdot)}(v;E)$ such that $u_j \to u$ in $L^{p(\cdot)}(v;E)$ and $\mathbf{g}_j \to \mathbf{g}$ in $\mathcal{L}_Q^{p(\cdot)}(E)$. We must show that $(u,\mathbf{g}) \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$, i.e. that $\int_E u(x)v(x)dx = 0$. Since for all j, $(u_j,\mathbf{g}_j) \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$, we have that $\int_E u_j(x)v(x)dx = 0$.

Thus,

$$\begin{aligned}
\left|\int_E u(x)v(x)dx\right| &= \left|\int_E (u(x)-u_j(x))v(x)dx\right| && (u_j \in \tilde{H}_Q^{1,p(\cdot)}(v;E)) \\
&\leq \int_E |u(x)-u_j(x)|v(x)dx \\
&\leq K_{p(\cdot)}\|(u-u_j)v\|_{L^{p(\cdot)}(E)}\|1\|_{L^{p'(\cdot)}(E)} && \text{(theorem (13))}
\end{aligned}$$

Since E is bounded, $\|1\|_{L^{p'(\cdot)}(E)} < \infty$. To see this, choose $\mu = |E \backslash E_\infty| + 1$. Then

$$\begin{aligned}
\rho_{p'(\cdot)}(1/\mu) &= \int_{E\backslash E_\infty} \left(\frac{1}{\mu}\right)^{p'(x)} dx + \|1/\mu\|_{L^\infty(E_\infty)} \\
&\leq \int_{E\backslash E_\infty} \frac{1}{\mu} dx + \|1/\mu\|_{L^\infty(E_\infty)} && (\lambda > 1) \\
&\leq \frac{1}{\mu}|E\backslash E_\infty| + \frac{1}{\mu} \\
&= \frac{|E\backslash E_\infty|+1}{\mu} \\
&= 1 && (\text{choice of } \mu)
\end{aligned}$$

Since E is bounded, $\mu = |E\backslash E_\infty| + 1 < \infty$ and so $\|1\|_{L^{p'(\cdot)}(E)} < \infty$. Since $u_j \to u$ in $L^{p(\cdot)}(v;E)$, we have that $\|(u-u_j)v\|_{L^{p(\cdot)}(E)} = \|u-u_j\|_{L^{p(\cdot)}(v;E)} \to 0$. Hence, $\int_E u(x)v(x)dx = 0$. Thus, $(u,\mathbf{g}) \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$. Therefore, $\tilde{H}_Q^{1,p(\cdot)}(v;E)$ is a closed subspace of the complete space $H_Q^{1,p(\cdot)}(v;E)$ and so is complete. Furthermore, this argument holds if $1 < p_- \leq p_+ < \infty$ and so $\tilde{H}_Q^{1,p(\cdot)}(v;E)$ is reflexive if $1 < p_- \leq p_+ < \infty$, since every closed subspace of a reflexive Banach space is reflexive. □

In [5], it was shown that given a bounded, open set $E \subset \mathbb{R}^n$, a constant exponent $1 <$

$p < \infty$, and a measurable matrix function $Q : E \to \mathcal{S}_n$, if $\gamma^{p/2} \in L^1_{loc}(E)$, then Q is p-Neumann if and only if Q has the Poincaré property of order p on E. We now translate the necessary definitions in [5] into their variable counterparts. We begin by defining the Poincaré property of order $p(\cdot)$.

Definition 36. (Poincaré Property of order $p(\cdot)$) Given a bounded open set $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$, a measurable matrix function $Q : E \to \mathcal{S}_n$ is said to have the Poincaré property of order $p(\cdot)$ on E if there is a positive constant $C_0 = C_0(E)$ such that for all $f \in C^1(\overline{E})$,

$$\|f - f_E\|_{L^{p(\cdot)}(v;E)} \leq C_0 \|\nabla f\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \tag{21}$$

For brevity, we will sometimes refer to the Poincaré property of order $p(\cdot)$ as the $p(\cdot)$-Poincaré property. Next, we define the $p(\cdot)$-Neumann property, and then define weak solutions.

Definition 37. ($p(\cdot)$-Neumann Property) Given a bounded, open set $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$, a measurable matrix function $Q : E \to \mathcal{S}_n$ is said to have the $p(\cdot)$-Neumann property on E if the following hold:

1. Given any $f \in L^{p(\cdot)}(v;E)$, there exists a weak solution $(u, \mathbf{g})_f \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$ to the weighted homogeneous Neumann problem

$$\begin{cases} \operatorname{div}\left(\left|\sqrt{Q(x)}\nabla u(x)\right|^{p(x)-2} Q(x)\nabla u(x)\right) & = |f(x)|^{p(x)-2} f(x)(v(x))^{p(x)} \text{ in } E \\ \mathbf{n}^t \cdot Q(x)\nabla u(x) & = 0 \text{ on } \partial E, \end{cases} \tag{22}$$

   where $\mathbf{n}$ is the outward unit normal vector of ∂E.

2. Any weak solution $(u,\mathbf{g})_f \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$ of (22) is regular: that is, there is a positive constant $C_{p(\cdot)} = C_{p(\cdot)}(v,E)$ such that

$$\|u\|_{L^{p(\cdot)}(v;E)} \leq C_{p(\cdot)}\|f\|_{L^{p(\cdot)}(v;E)}. \tag{23}$$

Definition 38. (Weak solutions) Let $E \subseteq \mathbb{R}^n$ be a bounded open set and $p(\cdot) \in \mathscr{P}(E)$. Given $f \in L^{p(\cdot)}(v;E)$, we say that the pair $(u,\mathbf{g})_f \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$ is a weak solution to the weighted homogeneous Neumann problem (22) if for all test functions $\varphi \in C^1(\overline{E}) \cap \tilde{H}_Q^{1,p(\cdot)}(v;E)$,

$$\int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p(x)-2}(\nabla\varphi(x))^T Q(x)\mathbf{g}(x)dx = -\int_E |f(x)|^{p(x)-2}f(x)\varphi(x)(v(x))^{p(x)}dx.$$

With these definitions, we can explore whether the equivalence between the p-Neumann property and the p-Poincaré property holds in the variable exponent setting. We must determine whether the $p(\cdot)$-Poincaré property implies the $p(\cdot)$-Neumann property. In the following section, we prove that the $p(\cdot)$-Poincaré property implies the existence of weak solutions to (22).

4.2 $p(\cdot)$-Poincaré Implies Existence of Weak Solutions

To prove that if Q has the Poincaré property of order $p(\cdot)$ on E, then there exists a weak solution to the weighted homogeneous Neumann problem (22), we will use Minty's theorem from [14]. While Minty's theorem was stated in chapter 2.2, for ease of reading, we will again state Minty's theorem and remind the reader of some notation.

Given a reflexive Banach space $\mathscr{B}$ denote its dual space by $\mathscr{B}^*$. Given a functional $\alpha \in \mathscr{B}^*$, write its value at $\varphi \in \mathscr{B}$ as $\alpha(\varphi) = \langle \alpha, \varphi \rangle$. Thus, if $\beta : \mathscr{B} \to \mathscr{B}^*$ and $u \in \mathscr{B}$, then we have $\beta(u) \in \mathscr{B}^*$ and so its value at φ is denoted by $\beta(u)(\varphi) = \langle \beta(u), \varphi \rangle$.

Theorem 39. *(Minty's theorem, [14]) Let $\mathscr{B}$ be a reflexive, separable Banach space and fix $\Gamma \in \mathscr{B}^*$. Suppose that $\mathscr{T} : \mathscr{B} \to \mathscr{B}^*$ is a bounded operator that is:*

1. *Monotone: $\langle \mathscr{T}(u) - \mathscr{T}(\varphi), u - \varphi \rangle \geq 0$ for all $u, \varphi \in \mathscr{B}$;*
2. *Hemicontinuous: for $z \in \mathbb{R}$, the mapping $z \to \langle \mathscr{T}(u + z\varphi), \varphi \rangle$ is continuous for all $u, \varphi \in \mathscr{B}$;*
3. *Almost Coercive: there exists a constant $\lambda > 0$ so that $\langle \mathscr{T}(u), u \rangle > \langle \Gamma, u \rangle$ for any $u \in \mathscr{B}$ satisfying $\|u\|_{\mathscr{B}} > \lambda$.*

Then the set of $u \in \mathscr{B}$ such that $\mathscr{T}(u) = \Gamma$ is non-empty.

To apply Minty's theorem to prove the existence of a weak solution, let $\mathscr{B} = \tilde{H}_Q^{1,p(\cdot)}(v;E)$. We can then define the operators Γ and $\mathfrak{T}$ to be the two sides of the weak solution condition in definition 38.

Definition 40. Fix $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \leq p_+ < \infty$. Let $f \in L^{p(\cdot)}(v;E)$. Define $\Gamma = \Gamma_f : \tilde{H}_Q^{1,p(\cdot)}(v;E) \to \mathbb{R}$ by for $\mathbf{w} = (w, \mathbf{h}) \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$,

$$\langle \Gamma, \mathbf{w} \rangle = -\int_E |f(x)|^{p(x)-2} f(x) w(x) (v(x))^{p(x)} dx.$$

Note that Γ_f is dependent on the given $f \in L^{p(\cdot)}(v;E)$. For ease of notation, we will simply write Γ when f is understood in the context.

Definition 41. Define $\mathfrak{T} : \tilde{H}_Q^{1,p(\cdot)}(v;E) \to \left(\tilde{H}_Q^{1,p(\cdot)}(v;E)\right)^*$ by for $\mathbf{u} = (u,\mathbf{g})$ and $\mathbf{w} = (w,\mathbf{h})$, elements of $\tilde{H}_Q^{1,p(\cdot)}(v;E)$,

$$\langle \mathfrak{T}(\mathbf{u}), \mathbf{w}\rangle = \int_E \left|\sqrt{Q(x)}\mathbf{g}(x)\right|^{p(x)-2} \mathbf{h}^t(x)Q(x)\mathbf{g}(x)dx$$

We now show that the operators Γ_f and $\mathfrak{T}$ satisfy the assumptions of Minty's theorem. To do so, we will often use theorem 18 from chapter 3. We begin by showing for all $f \in L^{p(\cdot)}(v;E)$, $\Gamma_f \in H_Q^{1,p(\cdot)}(v;E)^*$.

Lemma 42. *Let $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \leq p_+ < \infty$. Then for each $f \in L^{p(\cdot)}(v;E)$, $\Gamma = \Gamma_f$ is a bounded, linear functional on $H_Q^{1,p(\cdot)}(v;E)$..*

Proof. Let $f \in L^{p(\cdot)}(v;E)$. We will show that $\Gamma \in \left(\tilde{H}_Q^{1,p(\cdot)}(v;E)\right)^*$, i.e. that Γ is a bounded, linear functional. We first show that Γ is linear. Let $\mathbf{w} = (w,\mathbf{h}) \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$. Observe that for all $\alpha \in \mathbb{R}$,

$$\begin{aligned}
\langle \Gamma, \alpha\mathbf{w}\rangle &= -\int_E |f(x)|^{p(x)-2} f(x)\alpha w(x)(v(x))^{p(x)}dx \\
&= -\alpha \int_E |f(x)|^{p(x)-2} f(x)w(x)(v(x))^{p(x)}dx \\
&= \alpha\langle \Gamma, \mathbf{w}\rangle
\end{aligned}$$

Thus, Γ is linear. To show that Γ is bounded, it suffices to show that there exists a positive constant $C = C(f)$ such that

$$|\langle \Gamma, \mathbf{w}\rangle \leq C\|wv\|_{p(\cdot)}$$

since $\|wv\|_{p(\cdot)} = \|w\|_{L^{p(\cdot)}(v;E)} \leq \|\mathbf{w}\|_{\tilde{H}_Q^{1,p(\cdot)}(v;E)}$. Observe that by Holder's inequality in theorem 13, we have

$$\begin{aligned}
|\langle \Gamma, \mathbf{w} \rangle| &= \left| -\int_E |f(x)|^{p(x)-2} f(x) w(x) (v(x))^{p(x)} dx \right| \\
&\leq \int_E |f(x)|^{p(x)-1} (v(x))^{p(x)-1} w(x) v(x) dx \qquad \left(\left| \int f(x) dx \right| \leq \int |f(x)| dx \right) \\
&\leq K_{p(\cdot)} \| |f|^{p(\cdot)-1} v^{p(\cdot)-1} \|_{p'(\cdot)} \|wv\|_{p(\cdot)} \qquad \text{(theorem 13)}
\end{aligned}$$

Since $f \in L^{p(\cdot)}(v;E)$, we have that $\|fv\|_{p(\cdot)} < \infty$. By assumption, $1 < p_- \leq p_+ < \infty$, and so by theorem 18 $\| |f|^{p(\cdot)-1} v^{p(\cdot)-1} \|_{p'(\cdot)} < \infty$ as well. Choosing $C = K_{p(\cdot)} \| |f|^{p(\cdot)-1} v^{p(\cdot)-1} \|_{p'(\cdot)}$, we have that

$$|\langle \Gamma, \mathbf{w} \rangle \leq C \|wv\|_{p(\cdot)}$$

and so Γ is bounded, and hence $\Gamma \in \left(\tilde{H}_Q^{1,p(\cdot)}(v;E) \right)^*$ $\square$

In section 2.2, we presented the proof from [5] showing that the operator $\mathcal{T}$ is bounded in the constant exponent setting. We now move on to proving that, in the variable exponent setting, $\mathcal{T}$ is bounded, monotone, hemicontinuous. The approaches used in these proofs, mirror the methods used in [5].

Lemma 43. *Let $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \leq p_+ < \infty$. Then $\mathcal{T}$ is bounded*

Proof. We will show that $\mathcal{T}$ is bounded by showing the operator norm of $\mathcal{T}$ is uniformly bounded. Denote that operator norm by $\|\cdot\|_{op}$ and the linear functional norm by $|\cdot|_{op}$. Then the operator norm of $\mathcal{T} : \tilde{H}_Q^{1,p(\cdot)}(v;E) \to \left(\tilde{H}_Q^{1,p(\cdot)}(v;E) \right)^*$ is given by

$$\|\mathcal{T}\|_{op} = \sup\{ |\mathcal{T}(\mathbf{u})|_{op} : \|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} = 1 \}$$

where $|\mathcal{T}(\mathbf{u})|_{op} = \sup\{|\langle \mathcal{T}(\mathbf{u}), \mathbf{w}\rangle| : \|\mathbf{w}\|_{H_Q^{1,p(\cdot)}(v;E)} = 1\}$. Thus, it suffices to show that there exists a constant $C > 0$ such that for all $\mathbf{u}, \mathbf{w} \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$ with $\|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} = \|\mathbf{w}\|_{H_Q^{1,p(\cdot)}(v;E)} = 1$,

$$|\langle \mathcal{T}(\mathbf{u}), \mathbf{w}\rangle| \leq C.$$

Fix $\mathbf{u} = (u, \mathbf{g})$ and $\mathbf{w} = (w, \mathbf{h})$ in $\tilde{H}_Q^{1,p(\cdot)}(v;E)$ with $\|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} = \|\mathbf{w}\|_{H_Q^{1,p(\cdot)}(v;E)}$. Observe that by the Cauchy-Schwartz inequality on $\mathbb{R}^n$,

$$|\mathbf{h}^T Q\mathbf{g}| = |(\sqrt{Q}\mathbf{h})^T(\sqrt{Q}\mathbf{g})| \leq |\sqrt{Q}\mathbf{h}||\sqrt{Q}\mathbf{g}|.$$

Note that if $f \in L^{p(\cdot)}(v;E)$, then by theorem 18,

$$\||f|^{p(\cdot)-1}\|_{p'(\cdot)} \leq \|f\|_{p(\cdot)}^{p_*-1} \tag{24}$$

where $p_* = \begin{cases} p_- & \text{if } 0 < \|f\|_{p(\cdot)} < 1 \\ p_+ & \text{if } 1 \leq \|f\|_{p(\cdot)} < \infty \end{cases}$. Now observe that

$$\begin{aligned}
|\langle \mathcal{T}(\mathbf{u}), \mathbf{w}\rangle| &= \left| \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p(x)-2} (\mathbf{h}(x))^T Q(x)\mathbf{g}(x)dx \right| \\
&\leq \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p(x)-2} |(\mathbf{h}(x))^T Q(x)\mathbf{g}(x)|dx && \left(\left|\int f dx\right| \leq \int |f|dx\right) \\
&\leq \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p(x)-1} |\sqrt{Q(x)}\mathbf{h}(x)|dx && \text{(Cauchy-Schwartz)} \\
&\leq K_{p(\cdot)} \||\sqrt{Q}\mathbf{g}|^{p(\cdot)-1}\|_{p'(\cdot)} \||\sqrt{Q}\mathbf{h}\|_{p(\cdot)} && \text{(theorem 13)} \\
&\leq K_{p(\cdot)} \||\sqrt{Q}\mathbf{g}|\|_{p(\cdot)}^{p_*-1} \||\sqrt{Q}\mathbf{h}|\|_{p(\cdot)} && \text{(ineq. (24))} \\
&= K_{p(\cdot)} \|\mathbf{g}\|^{p_*-1}_{\mathcal{L}^{p(\cdot)}_Q(E)} \|\mathbf{h}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} && (\||\sqrt{Q}\mathbf{g}|\|_{p(\cdot)} = \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)}) \\
&\leq K_{p(\cdot)} \|\mathbf{u}\|^{p_*-1}_{H^{1,p(\cdot)}_Q(v;E)} \|\mathbf{w}\|_{H^{1,p(\cdot)}_Q(v;E)} && (\|\mathbf{u}\|_{H^{1,p(\cdot)}_Q(v;E)} \text{ norm def.}) \\
&= K_{p(\cdot)} && \text{(Choice of } \mathbf{u}, \mathbf{w})
\end{aligned}$$

Thus, $\mathcal{T}$ is bounded. □

Lemma 44. *Let $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \leq p_+ < \infty$. Then $\mathcal{T}$ is Monotone.*

Proof. Let $\mathbf{u} = (u, \mathbf{g})$ and $\mathbf{w} = (w, \mathbf{h})$ be in $\tilde{H}_Q^{1,p(\cdot)}(v; E)$. Observe that

$$
\begin{aligned}
&\langle \mathcal{T}(\mathbf{u}) - \mathcal{T}(\mathbf{w}), \mathbf{u} - \mathbf{w} \rangle \\
&= \langle \mathcal{T}(\mathbf{u}), \mathbf{u} - \mathbf{w} \rangle - \langle \mathcal{T}(\mathbf{w}), \mathbf{u} - \mathbf{w} \rangle \\
&= \int_E |\sqrt{Q}\mathbf{g}|^{p(\cdot)-2} (\mathbf{g} - \mathbf{h})^T Q \mathbf{g} - |\sqrt{Q}\mathbf{h}|^{p(\cdot)-2} (\mathbf{g} - \mathbf{h})^T Q \mathbf{h} dx \\
&= \int_E (\sqrt{Q}(\mathbf{g} - \mathbf{h}))^T (|\sqrt{Q}\mathbf{g}|^{p(\cdot)-2} \sqrt{Q}\mathbf{h}) - (\sqrt{Q}(\mathbf{g} - \mathbf{h}))^T (|\sqrt{Q}\mathbf{h}|^{p(\cdot)-2} \sqrt{Q}\mathbf{h}) dx \\
&= \int_E (\sqrt{Q}(\mathbf{g} - \mathbf{h}))^T \left[|\sqrt{Q}\mathbf{g}|^{p(\cdot)-2} \sqrt{Q}\mathbf{g} - |\sqrt{Q}\mathbf{h}|^{p(\cdot)-2} \sqrt{Q}\mathbf{h} \right] dx \\
&= \int_E \langle |\sqrt{Q}\mathbf{g}|^{p(\cdot)-2} \sqrt{Q}\mathbf{g} - |\sqrt{Q}\mathbf{h}|^{p(\cdot)-2} \sqrt{Q}\mathbf{h}, \sqrt{Q}\mathbf{g} - \sqrt{Q}\mathbf{h} \rangle_{\mathbb{R}^n} dx
\end{aligned}
$$

where the $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denotes the inner product on $\mathbb{R}^n$. Note that in the integrand we have suppressed the dependency on x. Observe that for each $x \in E$, the integrand is of the form

$$
\langle |s|^{p-2} s - |r|^{p-2} r, s - r \rangle_{\mathbb{R}^n}
$$

where $s, r \in \mathbb{R}^n$ and $p > 1$. For such p, s, r, an inequality in [7, chapter 10] shows this expression is nonnegative. Thus $\mathcal{T}$ is monotone. □

Before proving that $\mathcal{T}$ is hemicontinuous, we present an established result showing a connection between the modular, the norm, and the exponent function.

Proposition 45. *[4, Proposition 2.12] Given $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$, then the property that $f \in L^{p(\cdot)}(E)$ if and only if $\rho(f) < \infty$ is equivalent to assuming that $p_- = \infty$ or $p_+(E \backslash E_\infty) < \infty$.*

Lemma 46. *Let $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \leq p_+ < \infty$. Then $\mathcal{T}$ is hemicontinuous.*

Proof. Let $z, y \in \mathbb{R}$. Let $\mathbf{u} = (u, \mathbf{g})$ and $\mathbf{w} = (w, \mathbf{h})$ be in $\tilde{H}_Q^{1,p(\cdot)}(v;E)$. Define $\psi = \mathbf{g} + z\mathbf{h}$ and $\gamma = \mathbf{g} + y\mathbf{h}$. Then

$$\begin{aligned}
&\langle \mathcal{T}(\mathbf{u} + z\mathbf{w}) - \mathcal{T}(\mathbf{u}u + y\mathbf{w}), \mathbf{w}\rangle \\
&= \int_E |\sqrt{Q}\psi|^{p(\cdot)-2}\mathbf{h}^T Q\psi - |\sqrt{Q}\gamma|^{p(\cdot)-2}\mathbf{h}^T Q\gamma dx \\
&= \int_E (\sqrt{Q}\mathbf{h})^T \left[|\sqrt{Q}\psi|^{p(\cdot)-2}\sqrt{Q}\psi - |\sqrt{Q}\gamma|^{p(\cdot)-2}\sqrt{Q}\gamma \right] dx \\
&= \int_E (\sqrt{Q}\mathbf{h})^T \left[|\mathbf{r}|^{p(\cdot)-2}\mathbf{r} - |\mathbf{s}|^{p(\cdot)-2}\mathbf{s} \right] dx \qquad (25)
\end{aligned}$$

where $\mathbf{r} = \sqrt{Q}\psi$ and $\mathbf{s} = \sqrt{Q}\gamma$. Define $E^+ = \{x \in E : p(x) > 2\}$ and $E^- = \{x \in E : p(x) \leq 2\}$. We will show that the integral in equation (25) tends to 0 as $z \to y$ by considering the integral over E^+ and E^- separately. Observe that our choice of $\mathbf{r}, \mathbf{s}$ gives

$$\mathbf{r} - \mathbf{s} = \sqrt{Q}(\psi - \gamma) = \sqrt{Q}(z\mathbf{h} - y\mathbf{h}) = (z - y)\sqrt{Q}\mathbf{h}. \qquad (26)$$

Hence,

$$\| |\mathbf{r} - \mathbf{s}| \|_{p(\cdot)} = |z - y| \left\| |\sqrt{Q}\mathbf{h}| \right\|_{p(\cdot)} \leq |z - y| \|\mathbf{w}\|_{H_Q^{1,p(\cdot)}(v;E)} \qquad (27)$$

Furthermore, from [7, chapter 10], we have for $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n$ and $p > 2$,

$$\left| |\mathbf{r}|^{p-2}\mathbf{r} - |\mathbf{s}|^{p-2}\mathbf{s} \right| \leq (p-1)|\mathbf{r} - \mathbf{s}|(|\mathbf{s}|^{p-2} + |\mathbf{r}|^{p-2})$$

Combining this inequality with equation (26) and applying them to equation (25) over E^+, we have that

$$\begin{aligned}
&\left|\int_{E^+}(\sqrt{Q}\mathbf{h})^T\left[|\mathbf{r}|^{p(x)-2}\mathbf{r}-|\mathbf{s}|^{p(x)-2}\mathbf{s}\right]dx\right| \\
&\leq \int_{E^+}|\sqrt{Q}\mathbf{h}||p(x)-1||\mathbf{r}-\mathbf{s}|(|\mathbf{s}|^{p(x)-2}+|\mathbf{r}|^{p(x)-2})dx \\
&\leq |z-y||p_+-1|\int_{E^+}|\sqrt{Q}\mathbf{h}|^2\left||\mathbf{s}|^{p(x)-2}+|\mathbf{r}|^{p(x)-2}\right|dx
\end{aligned} \tag{28}$$

Since $p(x)>2$ on E^+, by Holder's inequality from theorem 13 with exponents $\frac{p(\cdot)}{2},\frac{p(\cdot)}{p(\cdot)-2}>1$, we have

$$\int_{E^+}|\sqrt{Q}\mathbf{h}|^2\left||\mathbf{s}|^{p(\cdot)-2}+|\mathbf{r}|^{p(\cdot)-2}\right|dx\leq K_{p(\cdot)/2}\left\||\sqrt{Q}\mathbf{h}|^2\right\|_{p(\cdot)/2}\left\||\mathbf{s}|^{p(\cdot)-2}+|\mathbf{r}|^{p(\cdot)-2}\right\|_{p(\cdot)/(p(\cdot)-2)}$$

where the norms are taken over the domain E^+. Since $p_+<\infty$, by proposition 45 $\||\sqrt{Q}\mathbf{h}|\|_{p(\cdot)}$ is finite if and only if $\int_E|\sqrt{Q}\mathbf{h}|^{p(x)}dx<\infty$. Since $\mathbf{h}\in\mathcal{L}^{p(\cdot)}_Q(E)$ and $p_+<\infty$, we have

$$\int_{E^+}\left||\sqrt{Q}\mathbf{h}|^2\right|^{p(x)/2}dx=\int_{E^+}|\sqrt{Q}\mathbf{h}|^{p(x)}dx<\infty$$

Thus, $\||\sqrt{Q}\mathbf{h}|^2\|_{p(\cdot)/2}<\infty$. Now by the triangle inequality,

$$\left\||\mathbf{s}|^{p(\cdot)-2}+|\mathbf{r}|^{p(\cdot)-2}\right\|_{p(\cdot)/(p(\cdot)-2)}\leq\left\||\mathbf{s}|^{p(\cdot)-2}\right\|_{p(\cdot)/(p(\cdot)-2)}+\left\||\mathbf{r}|^{p(\cdot)-2}\right\|_{p(\cdot)/(p(\cdot)-2)}.$$

Observe that

$$||\mathbf{s}|^{p(\cdot)-2}|^{p(\cdot)/(p(\cdot)-2)}=|\mathbf{s}|^{p(\cdot)}=|\sqrt{Q}\gamma|^{p(\cdot)}=|\sqrt{Q}(\mathbf{g}+y\mathbf{h})|^{p(\cdot)}.$$

Since $\mathbf{g}, \mathbf{h} \in \mathcal{L}_Q^{p(\cdot)}(E)$, so is $(\mathbf{g}+y\mathbf{h})$ and thus $\left\| |\sqrt{Q}(\mathbf{g}+y\mathbf{h})| \right\|_{p(\cdot)} < \infty$. Since $p_+ < \infty$, by proposition 45, we have

$$\int_{E^+} \left| |\sqrt{Q}(\mathbf{g}+y\mathbf{h})|^{p(x)-2} \right|^{p(x)/(p(x)-2)} dx = \int_{E^+} |\sqrt{Q}(\mathbf{g}+y\mathbf{h})|^{p(x)} dx < \infty.$$

Thus $\left\| |\mathbf{s}|^{p(\cdot)-2} \right\|_{p(\cdot)/(p(\cdot)-2)}$ is finite. The same argument shows $\left\| |\mathbf{r}|^{p(\cdot)-2} \right\|_{p(\cdot)/(p(\cdot)-2)}$ is finite. Therefore $\int_{E^+} |\sqrt{Q}\mathbf{h}|^2 \left| |\mathbf{s}|^{p(x)-2} + |\mathbf{r}|^{p(x)-2} \right| dx$ is finite, and so (28) converges to 0 as $z \to y$.

Now consider the domain E^-. Since $1 < p_- \leq p_+ < \infty$, we have $1 < p(x) \leq 2$ for all $x \in E^-$. From [7, Chapter 10], we have for $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n$ and $1 < p \leq 2$,

$$\left| |\mathbf{s}|^{p-2}\mathbf{s} - |\mathbf{r}|^{p-2}\mathbf{r} \right| \leq C(p)|\mathbf{s}-\mathbf{r}|^{p-1} \tag{29}$$

Since $p_- > 1$, and the constant $C(p)$ is finite for all $1 < p \leq 2$, then $C = \sup_{x \in E^-} C(p(x))$ is finite. Applying this and Holder's inequality from theorem 13 to equation (25), we get

$$\begin{aligned}
&\left| \int_{E^-} (\sqrt{Q}\mathbf{h})^T \left[|\mathbf{r}|^{p(x)-2}\mathbf{r} - |\mathbf{s}|^{p(x)-2}\mathbf{s} \right] dx \right| \\
&\leq \int_{E^-} |\sqrt{Q}\mathbf{h}| \left| |\mathbf{r}|^{p(x)-2}\mathbf{r} - |\mathbf{s}|^{p(x)-2}\mathbf{s} \right| dx \\
&\leq \int_{E^-} |\sqrt{Q}\mathbf{h}| C(p(x)) |\mathbf{s}-\mathbf{r}|^{p(x)-1} dx && \text{(ineq. (29))} \\
&\leq C \int_{E^-} |\sqrt{Q}\mathbf{h}| |\mathbf{s}-\mathbf{r}|^{p(x)-1} dx \\
&\leq C K_{p(\cdot)} \|\mathbf{h}\|_{\mathcal{L}_Q^{p(\cdot)}(E^-)} \left\| |\mathbf{s}-\mathbf{r}|^{p(\cdot)-1} \right\|_{p'(\cdot)} && \text{(theorem 13)} \\
&\leq C K_{p(\cdot)} \|\mathbf{h}\|_{\mathcal{L}_Q^{p(\cdot)}(E^-)} \| |\mathbf{s}-\mathbf{r}| \|_{p(\cdot)}^{p_*-1} && \text{(theorem 18)} \\
&\leq C K_{p(\cdot)} \|\mathbf{h}\|_{\mathcal{L}_Q^{p(\cdot)}(E^-)} (|z-y| \|\mathbf{w}\|_{p(\cdot)})^{p_*-1} && \text{(ineq. (27))}
\end{aligned}$$

where $p_* = \begin{cases} p_- & \text{if } 0 < |||\mathbf{s}-\mathbf{r}|||_{p(\cdot)} < 1 \\ p_+ & \text{if } 1 \leq |||\mathbf{s}-\mathbf{r}|||_{p(\cdot)} < \infty \end{cases}$. Since $p_+ < \infty$, we have $1 < p_* < \infty$. Thus, (25) converges to 0 on E^- as $z \to y$, and so $\mathcal{T}$ is hemicontinuous. □

Lastly, we will show that $\mathcal{T}$ is almost coercive. The following propositions will be used to do so.

Proposition 47. *[4, Corollary 2.23] Given $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(E)$, suppose $|E_\infty| = 0$. If $\|f\|_{p(\cdot)} > 1$ then*

$$\rho(f)^{1/p_+} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_-}.$$

If $0 < \|f\|_{p(\cdot)} \leq 1$, then

$$\rho(f)^{1/p_-} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_+}.$$

Proposition 48. *Given a measurable matrix function $Q : E \to \mathcal{S}_n$ and $p(\cdot) \in \mathscr{P}(E)$ with $p_+ < \infty$, the set $C^1(\overline{E}) \cap \tilde{H}_Q^{1,p(\cdot)}(v;E)$ is dense in $\tilde{H}_Q^{1,p(\cdot)}(v;E)$.*

Proof. Fix $(u,\mathbf{g}) \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$. By the definition of $H_Q^{1,p(\cdot)}(v;E)$, $C^1(\overline{E})$ is dense in $\tilde{H}_Q^{1,p(\cdot)}(v;E) \subseteq H_Q^{1,p(\cdot)}(v;E)$. Thus, there exists a sequence of functions $u_k \in C^1(\overline{E})$ such that $(u_k, \nabla u_k) \to (u,\mathbf{g})$ in norm. Let $y_k = u_k - (u_k)_E \in C^1(\overline{E}) \cap \tilde{H}_Q^{1,p(\cdot)}(v;E)$, where $(u_k)_E = \dfrac{1}{v(E)} \displaystyle\int_E u_k(x)v(x)dx$.

Then $\nabla y_k = \nabla u_k$, and so to prove $(y_k, \nabla y_k) \to (u, \mathbf{g})$ it suffices to show $(u_k - y_k) \to 0$ in $L^{p(\cdot)}(v;E)$. Since $(u, \mathbf{g}) \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$, we have $u_E = \int_E u(x)v(x)dx = 0$, and so

$$\begin{aligned}
\|u_k - y_k\|_{L^{p(\cdot)}(v;E)} &= \|u_k - (u_k - (u_k)_E)\|_{L^{p(\cdot)}(v;E)} \\
&= \|(u_k)_E - u_E\|_{L^{p(\cdot)}(v;E)} && (u_E = 0) \\
&= \|\frac{1}{v(E)}\int_E (u_k - u)v dx\|_{L^{p(\cdot)}(v;E)} && (u_E, (u_k)_E \text{ def.}) \\
&= \left|\frac{1}{v(E)}\right| \left|\int_E (u_k - u)v dx\right| \|1\|_{L^{p(\cdot)}(v;E)} && (\text{norm property}) \\
&\le \left|\frac{1}{v(E)}\right| \int_E |u_k - u| v dx \|1\|_{L^{p(\cdot)}(v;E)} \\
&\le \left|\frac{1}{v(E)}\right| K_{p(\cdot)} \|u_k - u\|_{L^{p(\cdot)}(v;E)} \|1\|_{L^{p'(\cdot)}(E)} \|1\|_{L^{p(\cdot)}(v;E)} && (\text{theorem } 13)
\end{aligned}$$

Since E is bounded, $1 < p_- \le p_+ < \infty$, and $v \in L^1_{loc}(E)$, then $\|1\|_{L^{p(\cdot)}(v;E)}$ and $\|1\|_{L^{p'(\cdot)}(E)}$ are finite. This is shown in the proof of theorem 35. Furthermore, since $u_k \to u$ in norm, $\|u_k - u\|_{L^{p(\cdot)}(v;E)} \to 0$ as $k \to \infty$. Thus $u_k - y_k \to 0$ in $L^{p(\cdot)}(v;E)$. $\square$

We are now ready to show that $\mathcal{T}$ is almost coercive with an additional assumption on the matrix function Q.

Lemma 49. *Let $E \subseteq \mathbb{R}^n$ be a bounded, open set, $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \le p_+ < \infty$, and $Q : E \to \mathcal{S}_n$ be a measurable matrix function. If Q has the Poincaré property of order $p(\cdot)$ on E, then for all $f \in L^{p(\cdot)}(v;E)$ in the definition of Γ, $\mathcal{T}$ is almost coercive.*

Proof. Let $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \le p_+ < \infty$. Assume Q has the Poincare property of order $p(\cdot)$ on E. Then for all $f \in C^1(\overline{E})$,

$$\|f - f_E\|_{L^{p(\cdot)}(v;E)} \le C_0 \|\nabla f\|_{\mathcal{L}_Q^{p(\cdot)}(E)}.$$

Then by proposition 48, for every $(u,\mathbf{g}) \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$, there exists a sequence of functions $\{u_k\}_{k=1}^{\infty} \subseteq C^1(\overline{E}) \cap \tilde{H}_Q^{1,p(\cdot)}(v;E)$ such that $(u_k, \nabla u_k) \to (u,\mathbf{g})$ in norm as $k \to \infty$. Thus, $\|u_k\|_{L^{p(\cdot)}(v;E)} \to \|u\|_{L^{p(\cdot)}(v;E)}$ and $\|\nabla u_k\|_{\mathcal{L}_Q^{p(\cdot)}(v;E)} \to \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(v;E)}$ as $k \to \infty$. Now since elements of $\tilde{H}_Q^{1,p(\cdot)}(v;E)$ have mean-zero, $u_E = 0$ and for all k, $(u_k)_E = 0$. Hence

$$\|u - u_E\|_{L^{p(\cdot)}(v;E)} = \lim_{k\to\infty} \|u_k - (u_k)_E\|_{L^{p(\cdot)}(v;E)} \le C_0 \lim_{k\to\infty} \|\nabla u_k\|_{\mathcal{L}_Q^{p(\cdot)}(v;E)} = C_0 \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(v;E)}.$$

Thus, the Poincare property of order $p(\cdot)$ holds for all $(u,\mathbf{g}) \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$. Fix $f \in L^{p(\cdot)}(v;E)$. Choose $\lambda > 1 + C_0$. Let $\mathbf{u} = (u,\mathbf{g}) \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$ such that $\|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} > \lambda$. Observe that

$$\begin{aligned}
\lambda &< \|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} \\
&= \|u\|_{L^{p(\cdot)}(v;E)} + \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \\
&\le (C_0 + 1)\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} && \text{(Poincaré)}
\end{aligned}$$

Since $\lambda > 1 + C_0$, we have that $\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} > 1$. Now observe that since $p_+ < \infty$, we have

$$\begin{aligned}
\langle \mathcal{T}(\mathbf{u}), \mathbf{u} \rangle &= \int_E |\sqrt{Q}\mathbf{g}|^{p(x)-2} \mathbf{g}^T Q \mathbf{g}\, dx \\
&= \int_E |\sqrt{Q}\mathbf{g}|^{p(x)} dx && (\mathbf{g}^T Q \mathbf{g} = |\sqrt{Q}\mathbf{g}|^2) \\
&\ge \|\mathbf{g}\|^{p_-}_{\mathcal{L}_Q^{p(\cdot)}(E)} && \text{(prop. 47)} \\
&\ge \frac{1}{C_0^{p_-}} \|u\|^{p_-}_{L^{p(\cdot)}(v;E)} && \text{(Poincaré)}
\end{aligned}$$

Consequently,

$$(C_0^{p_-}+1)\langle \mathcal{T}(\mathbf{u}),\mathbf{u}\rangle \geq \|\mathbf{g}\|^{p_-}_{\mathcal{L}_Q^{p(\cdot)}(E)} + \|u\|^{p_-}_{L^{p(\cdot)}(v;E)} = C(p_-)\|\mathbf{u}\|^{p_-}_{H_Q^{1,p(\cdot)}(v;E)}. \tag{30}$$

Now observe that

$$\begin{aligned}
|\langle \Gamma,\mathbf{u}\rangle| &= \left|-\int_E |f(x)|^{p(x)-2} f(x)u(x)(v(x))^{p(x)}dx\right| \\
&\leq \int_E |f(x)|^{p(x)-1}(v(x))^{p(x)-1}|u(x)|v(x)dx \\
&\leq K_{p(\cdot)}\|f^{p(\cdot)-1}v^{p(\cdot)-1}\|_{p'(\cdot)}\|u\|_{L^{p(\cdot)}(v;E)} && \text{(theorem 13)} \\
&\leq K_{p(\cdot)}\|f^{p(\cdot)-1}v^{p(\cdot)-1}\|_{p'(\cdot)}\|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} && (\|u\|_{L^{p(\cdot)}(v;E)} \leq \|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)})
\end{aligned}$$

Since $f \in L^{p(\cdot)}(v;E)$, $\|fv\|_{p(\cdot)} < \infty$. Moreover, since $1 < p_- \leq p_+ < \infty$, by theorem 18 $\|f^{p(\cdot)-1}v^{p(\cdot)-1}\|_{p'(\cdot)} < \infty$ as well. Letting $C(f) = K_{p(\cdot)}\|f^{p(\cdot)-1}v^{p(\cdot)-1}\|_{p'(\cdot)}$, we have

$$\begin{aligned}
C(f)\|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} &= C(f)\|\mathbf{u}\|^{1-p_-}_{H_Q^{1,p(\cdot)}(v;E)}\|\mathbf{u}\|^{p_-}_{H_Q^{1,p(\cdot)}(v;E)} \\
&\leq C(f)\|\mathbf{u}\|^{1-p_-}_{H_Q^{1,p(\cdot)}(v;E)}\frac{C_0^{p_-}+1}{C(p_-)}\langle \mathcal{T}(\mathbf{u}),\mathbf{u}\rangle
\end{aligned}$$

Defining the constant $\mathscr{C} = C(f)\frac{C_0^{p_-}+1}{C(p_-)}$, we have

$$|\langle \Gamma,\mathbf{u}\rangle| \leq \mathscr{C}\|\mathbf{u}\|^{1-p_-}_{H_Q^{1,p(\cdot)}(v;E)}\langle \mathcal{T}(\mathbf{u}),\mathbf{u}\rangle$$

Thus, if we further assume $\lambda > \mathscr{C}^{1/(p_--1)}$ and $\|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} > \lambda$, then $\mathscr{C}^{1/(p_--1)} < \|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)}$. This in turn implies $\mathscr{C}\|\mathbf{u}\|^{1-p_-}_{H_Q^{1,p(\cdot)}(v;E)} < 1$. Therefore, if $\lambda > 1 + C_0$ and

$\lambda > \mathscr{C}^{1/(p_- - 1)}$, then

$$|\langle \Gamma, \mathbf{u} \rangle| < \langle \mathcal{T}(\mathbf{u}), \mathbf{u} \rangle$$

and hence, $\mathcal{T}$ is almost coercive. □

We have satisfied all the hypotheses of Minty's theorem 39. Hence, we have proven that if Q has the Poincaré property of order $p(\cdot)$ on E, then the existence condition of definition 37 is satisfied. Thus, $p(\cdot)$-Poincaré implies existence of weak solutions to the weighted homogeneous Neumann problem. It remains to be determined whether every weak solution is regular. It also remains to be determined whether $p(\cdot)$-Neumann implies $p(\cdot)$-Poincaré. In the following section, we show that these remaining pieces might not hold as desired.

4.3 Doubts on Desired Equivalence

In exploring the validity of the regularity condition (23), we will encounter some problems. These problems arise when applying the proposition 47, and the conjugate norm relationship in theorem 18. The problem comes down to a major difference between the constant exponent setting and the variable exponent setting. In the variable exponent setting, the supremum and infimum of the exponent function are allowed to differ, where as in the constant exponent setting, the supremum and infimum of the exponent are the same. This difference leads to the appearance of exponents in the regularity inequality (23). This is demonstrated in the following theorem.

Theorem 50. *Let $E \subseteq \mathbb{R}^n$ be a bounded, open set, $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \leq p_+ < \infty$ and $Q : E \to \mathcal{S}_n$ be a measurable matrix function with the Poincaré property of order $p(\cdot)$ on E. Let $f \in L^{p(\cdot)}(v;E)$ and $(u, \mathbf{g})_f \in \tilde{H}^{1,p(\cdot)}_Q(v;E)$ be a weak solution to the weighted* *homogeneous Neumann problem* (22). *Define*

$$p_* = \begin{cases} p_+ & \text{if } \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} < 1 \\ p_- & \text{if } \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \geq 1 \end{cases} \qquad \text{and} \qquad r_* = \begin{cases} p_+ & \text{if } \|f\|_{L^{p(\cdot)}(v;E)} \geq 1 \\ p_- & \text{if } \|f\|_{L^{p(\cdot)}(v;E)} < 1 \end{cases}.$$

Then there is a constant $C = C(p(\cdot), E)$ *such that*

$$\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \leq C\|f\|_{L^{p(\cdot)}(v;E)}^{\frac{r_*-1}{p_*-1}}.$$

Proof. Observe that

$$\begin{aligned}
\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}^{p_*} &\leq \int_E |\sqrt{Q}\mathbf{g}|^{p(x)} dx && \text{(prop. 47)} \\
&= \int_E |\sqrt{Q}\mathbf{g}|^{p(x)-2} \mathbf{g}^T Q\mathbf{g} dx && (|\sqrt{Q}\mathbf{g}|^2 = \mathbf{g}^T Q\mathbf{g}) \\
&= -\int_E |f|^{p(x)-2} f u v^{p(x)} dx && \text{(def. 38)} \\
&\leq \int_E |f|^{p(x)-1} v^{p(x)-1} |u| v dx && (\alpha \leq |\alpha|) \\
&\leq K_{p(\cdot)} \|(fv)^{p(\cdot)-1}\|_{p'(\cdot)} \|uv\|_{p(\cdot)} && \text{(theorem 13)} \\
&\leq K_{p(\cdot)} C_0 \|(fv)^{p(\cdot)-1}\|_{p'(\cdot)} \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} && \text{(Poincaré)} \\
&\leq K_{p(\cdot)} C_0 \|fv\|_{p(\cdot)}^{r_*-1} \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} && \text{(theorem 18)} \\
&= K_{p(\cdot)} C_0 \|f\|_{L^{p(\cdot)}(v;E)}^{r_*-1} \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}
\end{aligned}$$

Thus, $\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}^{p_*-1} \leq K_{p(\cdot)} C_0 \|f\|_{L^{p(\cdot)}(v;E)}^{r_*-1}$, and so we have that

$$\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \leq (K_{p(\cdot)} C_0)^{1/(p_*-1)} \|f\|_{L^{p(\cdot)}(v;E)}^{(r_*-1)/(p_*-1)}.$$

Since p_* has only two possible values, it suffices to choose C to be the larger of the two possible values of $(K_{p(\cdot)}C_0)^{1/(p_*-1)}$. $\square$

Remark 51. By the Poincaré property of order $p(\cdot)$, we also have that $\|u\|_{L^{p(\cdot)}(v;E)} \leq C_0\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}$. Hence, in the case where $\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} < 1 \leq \|f\|_{L^{p(\cdot)}(v;E)}$ or $\|f\|_{L^{p(\cdot)}(v;E)} < 1 \leq \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}$, we have that $p_* = r_*$, and so we can achieve the desired regularity estimate by choosing the larger bounding constant. Otherwise, the exponent in the inequality is not 1 unless $p_- = p_+$, i.e. $p(\cdot)$ is a constant function. Thus, this result is consistent with the regularity condition in the constant exponent setting in [5].

It should be noted that theorem 50 does not necessarily disprove the conjecture that the $p(\cdot)$-Poincaré property is equivalent to the $p(\cdot)$-Neumann property. To disprove this conjecture requires a counter example. However, theorem 50 does call into question the validity of one direction of the desired equivalence. It is doubtful that the $p(\cdot)$-Poincaré property implies the $p(\cdot)$-Neumann property. This leads to the other direction of the desired equivalence. Does the $p(\cdot)$-Neumann property imply the $p(\cdot)$-Poincaré property? Unfortunately, we encounter similar problems when exploring this question. We begin the exploration of this question by establishing a useful lemma.

Lemma 52. *Let $E \subseteq \mathbb{R}^n$ be a bounded, open set, $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \leq p_+ < \infty$, and $Q : E \to \mathbb{S}_n$ be a measurable matrix function. Suppose Q has the $p(\cdot)$-Neumann property of order $p(\cdot)$ on E. Let $f \in L^{p(\cdot)}(v;E)$ and $(u,\mathbf{g})_f \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$ be a weak solution to the weighted homogeneous Neumann problem* (22). *Define*

$$p_* = \begin{cases} p_+ & \text{if } \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} < 1 \\ p_- & \text{if } \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \geq 1 \end{cases} \qquad \text{and} \qquad r_* = \begin{cases} p_+ & \text{if } \|f\|_{L^{p(\cdot)}(v;E)} \geq 1 \\ p_- & \text{if } \|f\|_{L^{p(\cdot)}(v;E)} < 1 \end{cases}.$$

Then there exists a constant $C = C(p(\cdot), v, E)$ *such that*

$$\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \leq C\|f\|_{L^{p(\cdot)}(v;E)}^{r_*/p_*}.$$

Proof. By proposition 47, $\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}^{p_*} \leq \rho_{p(\cdot)}(|\sqrt{Q}\mathbf{g}|)$. Thus,

$$\begin{aligned}
\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}^{p_*} &\leq \int_E |\sqrt{Q}\mathbf{g}|^{p(x)} dx && \text{(prop. 47)} \\
&= \int_E |\sqrt{Q}\mathbf{g}|^{p(x)-2} \mathbf{g}^T Q\mathbf{g} dx && (\mathbf{g}^T Q\mathbf{g} = |\sqrt{Q}\mathbf{g}|^2) \\
&= -\int_E |f|^{p(x)-2} f u v^{p(x)} dx && \text{(definition 38)} \\
&\leq \int_E |f|^{p(x)-1} v^{p(x)-1} |u| v dx && (\alpha \leq |\alpha|) \\
&\leq K_{p(\cdot)} \|(fv)^{p(\cdot)-1}\|_{p'(\cdot)} \|uv\|_{p(\cdot)} && \text{(theorem 13)} \\
&\leq K_{p(\cdot)} \|fv\|_{p(\cdot)}^{r_*-1} \|u\|_{L^{p(\cdot)}(v;E)} && \text{(theorem 18)} \\
&\leq K_{p(\cdot)} C_{p(\cdot)} \|f\|_{L^{p(\cdot)}(v;E)}^{r_*-1} \|f\|_{L^{p(\cdot)}(v;E)} && \text{(regularity (23))}
\end{aligned}$$

Thus, $\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}^{p_*} \leq C\|f\|_{L^{p(\cdot)}(v;E)}^{r_*}$, where $C = K_{p(\cdot)} C_{p(\cdot)}$ depends on $p(\cdot), v$ and E. Raising both sides of this inequality to the power of $1/p_*$ and choosing the larger of the two possible bounding constants yields the desired inequality. □

We now mention a useful proposition that will be key in starting inequalities in the proof of the theorem that follows.

Proposition 53. *([4, Proposition 2.21]) Let* $E \subseteq \mathbb{R}^n$ *and* $p(\cdot) \in \mathscr{P}(E)$. *Then for all nontrivial* $f \in L^{p(\cdot)}(E)$, $\rho(f/\|f\|_{p(\cdot)}) = 1$ *if and only if* $p_+(E/E_\infty) < \infty$.

We now state our conclusion that the $p(\cdot)$-Neumann property might not imply the Poincaré property of order $p(\cdot)$. As we will show, we get an inequality with exponents depending on f and its weak solution $(u,\mathbf{g})$.

Theorem 54. *Let $E \subseteq \mathbb{R}^n$ be a bounded, open set, $p(\cdot) \in \mathscr{P}(E)$ with $1 < p_- \le p_+ < \infty$, and $Q : E \to \mathbb{S}_n$ be a measurable matrix function. Suppose Q has the $p(\cdot)$-Neumann property on E. Let $f \in C^1(\overline{E})$ and $(u,\mathbf{g})_f \in \tilde{H}^{1,p(\cdot)}_Q(v;E)$ be a weak solution to* (22)*. Define p_*, r_*, p'_* and r'_* by*

$$p_* = \begin{cases} p_+ & \textit{if } \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} < 1 \\ p_- & \textit{if } \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} \ge 1 \end{cases}, \qquad r_* = \begin{cases} p_+ & \textit{if } \|f\|_{L^{p(\cdot)}(v;E)} \ge 1 \\ p_- & \textit{if } \|f\|_{L^{p(\cdot)}(v;E)} < 1 \end{cases},$$

$$p'_* = \begin{cases} p_- & \textit{if } \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} < 1 \\ p_+ & \textit{if } \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} \ge 1 \end{cases}, \qquad \textit{and} \qquad r'_* = \begin{cases} p_- & \textit{if } \|f\|_{L^{p(\cdot)}(v;E)} \ge 1 \\ p_+ & \textit{if } \|f\|_{L^{p(\cdot)}(v;E)} < 1 \end{cases}.$$

Then there is a constant $C = C(p(\cdot),v,E)$ such that

$$\|f\|_{L^{p(\cdot)}(v;E)}^{r'_* - \frac{r_*}{p_*}(p'_* - 1)} \le C\|\nabla f\|.$$

Proof. Since $p_+ < \infty$, we can use proposition 53. Observe that

$$\begin{aligned}
1 &= \rho_{p(\cdot)}\left(\frac{fv}{\|fv\|_{p(\cdot)}}\right) && \text{(prop. 53)}\\
&= \int_E \|fv\|_{p(\cdot)}^{-p(x)} |fv|^{p(x)} dx\\
&\le \|fv\|_{p(\cdot)}^{-r'_*} \int_E |f|^{p(x)} v^{p(x)} dx\\
&= \|fv\|_{p(\cdot)}^{-r'_*} \int_E |f|^{p(x)-2} f f v^{p(x)} dx && (f^2 = |f|^2)\\
&= -\|fv\|_{p(\cdot)}^{-r'_*} \int_E |\sqrt{Q}\mathbf{g}|^{p(x)-2} \nabla f^T Q\mathbf{g} dx && \text{(def. 38)}\\
&\le \|fv\|_{p(\cdot)}^{-r'_*} \int_E |\sqrt{Q}\mathbf{g}|^{p(x)-1} |\sqrt{Q}\nabla f| dx && (\alpha < |\alpha|)\\
&\le K_{p(\cdot)} \|fv\|_{p(\cdot)}^{-r'_*} \||\sqrt{Q}\mathbf{g}|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)} \||\sqrt{Q}\nabla f|\|_{L^{p(\cdot)}(E)} && \text{(theorem 13)}\\
&\le K_{p(\cdot)} \|f\|_{L^{p(\cdot)}(v;E)}^{-r'_*} \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}^{p'_*-1} \|\nabla f\|_{\mathcal{L}_Q^{p(\cdot)}(E)} && \text{(theorem 18)}
\end{aligned}$$

Thus, $\|f\|_{L^{p(\cdot)}(v;E)}^{r'_*} \le \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}^{p'_*-1} \|\nabla f\|_{\mathcal{L}_Q^{p(\cdot)}(E)}$. By lemma 52,

$$\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}^{p'_*-1} \le \|f\|_{L^{p(\cdot)}(v;E)}^{\frac{r_*}{p_*}(p'_*-1)}.$$

Applying this to our inequality and dividing appropriately yields the desired inequality.

□

Note that for all $f \in C^1(\overline{E})$, we have $f - f_E \in C^1(\overline{E})$ and $\nabla(f - f_E) = \nabla f$. Thus, if $\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} < 1 \le \|f\|_{L^{p(\cdot)}(v;E)}$ or $\|f\|_{L^{p(\cdot)}(v;E)} < 1 \le \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}$ then $r'_* - \frac{r_*}{p_*}(p'_* - 1) = 1$, giving the desire Poincaré inequality. Otherwise, the exponent does not reduce to 1 unless $p(\cdot) = p$ is a constant. In this case, $p_* = r_* = p'_* = r'_* = p$, and so

$$r'_* - \frac{r_*}{p_*}(p'_* - 1) = p - (p-1) = 1.$$

Hence, this result is consistent with theorem 1.3 in [5], where for $1 < p < \infty$, the p-Neumann Property is equivalent to the Poincaré property of order p.

However, if $p(\cdot)$ is not a constant, and $\|f\|_{L^{p(\cdot)}(v;E)}$ and $\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}$ are both at least 1 or both smaller than 1, then $\|f\|_{L^{p(\cdot)}(v;E)} \geq \|f\|_{L^{p(\cdot)}(v;E)}^{r'_* - \frac{r_*}{p_*}(p'_* - 1)}$. Consequently, in such cases we cannot improve our arguments to achieve the exact form of the desired Poincaré inequality in definition 36.

REFERENCES

[1] S.-K. Chua, S. Rodney, and R. Wheeden, *A compact embedding theorem for generalized Sobolev spaces*, Pacific J. Math. **265** (2013), no. 1, 17–57. MR 3095112

[2] D. Cruz-Uribe, K. Moen, and V. Naibo, *Regularity of solutions to degenerate p-Laplacian equations*, J. Math. Anal. Appl. **401** (2013), no. 1, 458–478. MR 3011287

[3] D. Cruz-Uribe, K. Moen, and S. Rodney, *Matrix $\mathscr{A}_p$ weights, degenerate Sobolev spaces, and mappings of finite distortion*, J. Geom. Anal. **26** (2016), no. 4, 2797–2830. MR 3544941

[4] D. V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Heidelberg, 2013, Foundations and harmonic analysis. MR 3026953

[5] E. R. David Cruz-Uribe, Scott Rodney, *Poincare inequalities and newmann problems for the p-laplacian*, (2017).

[6] E. B. Fabes, C. E. Kenig, and R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116. MR 643158

[7] P. Lindqvist, *Notes on the p-laplace equation*, (2006), Foundations and harmonic analysis.

[8] D. D. Monticelli, S. Rodney, and R. L. Wheeden, *Boundedness of weak solutions of degenerate quasilinear equations with rough coefficients*, Differential Integral Equations **25** (2012), no. 1-2, 143–200. MR 2906551

[9] ———, *Harnack's inequality and Hölder continuity for weak solutions of degenerate quasilinear equations with rough coefficients*, Nonlinear Anal. **126** (2015), 69–114.

MR 3388872

[10] D. D. Monticelli and S. Rodney, *Existence and spectral theory for weak solutions of Neumann and Dirichlet problems for linear degenerate elliptic operators with rough coefficients*, J. Differential Equations **259** (2015), no. 8, 4009–4044. MR 3369270

[11] A. Ron and Z. Shen, *Frames and stable bases for shift-invariant subspaces of* $L_2(\mathbf{R}^d)$, Canad. J. Math. **47** (1995), no. 5, 1051–1094. MR 1350650

[12] E. T. Sawyer and R. L. Wheeden, *Hölder continuity of weak solutions to subelliptic equations with rough coefficients*, Mem. Amer. Math. Soc. **180** (2006), no. 847, x+157. MR 2204824

[13] ———, *Degenerate Sobolev spaces and regularity of subelliptic equations*, Trans. Amer. Math. Soc. **362** (2010), no. 4, 1869–1906. MR 2574880

[14] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs, vol. 49, American Mathematical Society, Providence, RI, 1997. MR 1422252

VITA

Michael Penrod

EDUCATION

Master of Science student in Mathematics at Sam Houston State University, August 2018 – present. Thesis title: "Exploring the Equivalence Between the Poincaré Property of Order p and the p-Neumann Property in the Variable Exponent Setting."

Bachelor of Science (Dec. 2017) in Mathematics, Sam Houston State University, Huntsville, Texas.

ACADEMIC EMPLOYMENT

Graduate Teaching Assistant, Department of Mathematics and Statistics, Sam Houston State University, August 2018 - present. Responsible of delivering every lecture, and all other aspects of course management, such as, creating activities to monitor learning, grading and recording grades, suggesting solutions to assigned problems, addressing student performance issues, developing ways to improve learning and understanding, assigning final grades.

PRESENTATIONS AT PROFESSIONAL MEETINGS

Michael Penrod. Variable Lebesgue Spaces: A Comparison. Joint Mathematics Meetings, Denver, Colorado, Jan. 17th 2020

ACADEMIC AWARDS

Outstanding Graduate Student in Mathematics, Department of Mathematics and Statistics, Sam Houston State University, 2020

Outstanding Graduate Student in Mathematics, Department of Mathematics and Statistics, Sam Houston State University, 2019

Outstanding Senior Mathematics Major, Department of Mathematics and Statistics, Sam Houston State University, 2017

Academic Excellence Award, Department of Mathematics and Statistics, Sam Houston State University, 2017

Outstanding minor in Statistical Theory, Department of Mathematics and Statistics, Sam Houston State University, 2017

Outstanding Junior Mathematics Major, Department of Mathematics and Statistics, Sam Houston State University, 2016

Member of Pi Mu Epsilon