Math 2440 **WS #15 – The Fund. Theorem of Line Integrals** Name: ______________

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Earlier we defined a **vector field** as a function:

$$F : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{or } F : \mathbb{R}^3 \to \mathbb{R}^3$$

We defined a vector field $F$ to be **conservative** if there is a function $f$ so that $F = \nabla f$, and we call $f$ the **potential function** for the conservative vector field $F$.

We also defined the work of a force field $F = \langle M, N, P\rangle$ along a curve $C$ parameterized by $\overrightarrow{r}(t)$, $a \leq t \leq b$ to be

$$W = \int_C F \cdot \overrightarrow{T}\, ds = \int_a^b F \cdot \overrightarrow{r'}\, dt$$

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1. Suppose $F = \langle M, N, P\rangle$ is a *conservative* vector field and $C$ is a curve parameterized by $\overrightarrow{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \leq t \leq b$.

   (a) Since $F$ is conservative, there is....

   (b) We know that the work done by this conservative vector field along the curve $C$ can be found to be:

   $W = \displaystyle\int_C F \cdot \overrightarrow{T}\, ds$

   $= \displaystyle\int_a^b F \cdot \overrightarrow{r'}\, dt$

   $= \displaystyle\int_a^b \langle \quad , \quad , \quad \rangle \cdot \langle \quad , \quad , \quad \rangle\, dt$

   $= \displaystyle\int_a^b \langle \quad , \quad , \quad \rangle \cdot \langle \quad , \quad , \quad \rangle\, dt$ [because $F$ is conservative]

   $= \displaystyle\int_a^b \frac{d}{dt}[ \qquad ]\, dt$

   $=$ [because of the Fundamental Theorem of Calculus]

   (c) The Fundamental Theorem of Line Integrals:

   If $F$ is a *conservative* vector field with potential function $f$, and $C$ is a curve parameterized by $\overrightarrow{r}(t)$ for $a \leq t \leq b$ then the work done by $F$ along the curve $C$ may be found by

   $$W = \int_C F \cdot \overrightarrow{T}\, ds = \int_a^b F \cdot \overrightarrow{r'}\, dt =$$

2. Verify that $F$ is conservative and find its potential function. Then compute the work done by $F$ along the curve parameterized by $\vec{r}(t)$.

$$F = \langle ye^z,\, xe^z,\, xye^z \rangle \qquad\qquad \vec{r}(t) = \langle t^2,\, t^3,\, t-1 \rangle,\ 1 \leq t \leq 2$$

3. Suppose $C$ is some curve from the point $P = (1, 2, 3)$ to $Q = (0, 6, 2)$. What can you say about the work done by the vector field $F = \langle y,\, x,\, z^3 \rangle$ along this unknown curve $C$?

4. Suppose $F$ is some conservative vector field with potential function $f$. Also suppose $C_1$ and $C_2$ are two different curves which both start at the same point $P$ and end at the same point $Q$. What can you say about the work done by $F$ along these two different curves?

5. Complete the following sentence. (It's a first *corollary* to the Fundamental Theorem of Line Integrals.)

   If $F$ is a conservative vector field, then the work done by $F$ from point $P$ to point $Q$ ...

6. Suppose that $C$ is a special curve we will call *closed*. That is, $C$ is parameterized by $\overrightarrow{r}(t)$ for $a \leq t \leq b$ but we know $\overrightarrow{r}(a) = \overrightarrow{r}(b)$. What can you say about the work done by a conservative vector field $F$ along a *closed* curve $C$?

7. Complete the following sentence. (It's another *corollary* to the Fundamental Theorem of Line Integrals.)

   If $F$ is conservative vector field, then the work done by $F$ around a closed curve $C$ is ....

8. Suppose $F = \langle 2xyz^{-1}, z + x^2z^{-1}, y - x^2yz^{-2} \rangle$. Compute the work done by $F$ along each of the curves below.

   (a) The ellipse $3y^2 + 7z^2 = 10$, starting at $(0, 1, 1)$, going around three times.

   (b) The ellipse $3y^2 + 7z^2 = 10$, from $(1, 1)$ to $(0, 1, -1)$