JOURNAL OF INQUIRY-BASED LEARNING IN MATHEMATICS
No. **31**, (May 2013)

Differential Calculus

Brian M. Loft

Sam Houston State University

Contents

Introduction

These notes are intended to be used in a one semester course in differential calculus. Rather than having the structure of a typical textbook (lecture, examples, practice problems at home), each chapter consists of a carefully designed sequence of problems and questions that – if completely solved and understood – will deliberately lead each student to a full comprehension of the material of differential calculus.

This student-centered (as opposed to instructor-centered) instruction has proven to be highly effective at all levels of learning. Commonly referred to as Inquiry- or Discovery-Based Learning, this method was pioneered several decades ago at UT-Austin by R. L. Moore. A controversial figure, Dr. Moore championed the philosophy that the level to which students can learn mathematics should not be damped by the knowledge of the instructor. In other words, it should be possible for a student to learn more than the instructor knows. This style of teaching (or rather of learning) more than allows for this possibility.

Read each problem carefully. Some may be solved quickly in one or two minutes. Some may take several days. Do not move on to the next problem without having a complete understanding of every previous solution. These problems are designed to prompt discussion in the classroom that will reveal some of the biggest ideas in calculus. That is, attendance in class each day is essential for even the most basic understanding. In short, if you don't work on these problems several hours each week outside of class as well as attend each and every class, it will be impossible to receive a passing grade in this course.

As the instructor of this course and the author of these notes, I can promise you several things. First, this course will be very demanding. I expect quite a bit from my students in all of my courses, but this one in particular will require more time and effort than usual. Secondly, this effort will be rewarded. I can guarantee a more complete understanding of differential calculus to any student that does what is expected. And lastly, I promise that this class will be fun. This atypical method of learning fosters a collegial environment among student and instructor. The more time you spend working either alone or with classmates outside of class and in my office, the more you will get out of the semester. Have fun!!

These notes would not have been possible without generous funding and encouragement from the Educational Advancement Foundation and in particular the Academy of Inquiry-Based Learning. For more information, visit `http://www.inquirybasedlearning.org`. I encourage each student to spend twenty minutes viewing the three “Videos about IBL” accessible at this webpage.

Acknowledgments

The author would like to acknowledge the faculty of the SHSU Department of Mathematics and Statistics for their encouragement and advice during the two semesters of testing these notes. Special gratitude is extended to Dr. Martin Montgomery for the generous donation of his time in the use of the notes in his classroom.

The Educational Advancement Foundation provided the funding that was necessary for the development of these course notes. Thanks to them, I was able to find the time in Summer 2011 to write and edit. The Academy of Inquiry Based Learning is responsible for providing me the courage and motivation to start this project. Attending their annual *Legacy of R.L. Moore* meetings has proved to be invaluable to me and my career.

Chapter 0

Preliminaries

> *Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigour should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere.*
>
> – W. S. Anglin

Differential calculus may be thought of as the study of functions, how their values change, and their application to the natural world. Therefore, in order to learn calculus in any meaningful manner, we must have a firm understanding of functions and their properties. Throughout this semester, we will use calculus to analyze the properties and uses of polynomial, rational, trigonometric, exponential, and logarithmic functions.

In this introductory chapter, we will be reminded of the basic operations of functions, without concentrating on any one particular *type* of function. We will begin later chapters with a review of some of the important functions (trigonometric, exponential, logarithmic) that make calculus such a useful tool. For now, however, we will need only to examine the *absolute value function* and the definition of the *average rate of change* of a function over an interval.

While reading each chapter, it is important that each definition and example is written down to the point of complete understanding. When a problem is encountered, **a complete solution must be written down before moving on to the next problem**. These problems have been carefully designed to lead each student to a thorough understanding of all of the material.

Definition 1. *A* **function** *f is a set of ordered pairs (x,y) in the plane $\mathbb{R}^2$, no two of which are on the same vertical line. We often write $f(x) = y$ to indicate the particular ordered pair (x,y) in f. The collection of all real numbers x from the ordered pairs (x,y) is called the* **domain** *of the function,* *while the collection of all real numbers y is called the* **range** *of the function.*

One way to think of a function might be as a machine that takes a number from the domain and – using the rule of the function – "converts" it to a number in the range. Each time a particular number from the domain is put into this machine, the same number from the range is produced. It may be convenient to think of the domain of a function as the set of all possible "inputs" into this machine while the range is the collection of all "outputs."

Since a function is defined as a collection of ordered pairs of real numbers, and any ordered pair of real numbers may be considered a point in the x-y plane, there is no distinction between a function f and its **graph** in the plane.

Example 2. *The ordered pairs* $\{(1,7),(2,8),(3,9),(4,8)\}$ *define a function* f*, with*

$$f(1) = 7, \qquad f(2) = 8, \qquad f(3) = 9, \qquad f(4) = 8$$

Note that each of the four "inputs" has exactly one output. Also note that one output (in this case 8) could be the function value for more than one input. That is to say, one range element may correspond to more than one domain element.

Problem 3. *Do the ordered pairs* $\{(3,-3),(4,1),(5,0),(3,7)\}$ *define a function? Explain your answer.*

A quick remark on some notation that we will use throughout the semester: we use the symbol "$\in$" to denote the phrase "is an element of." So if x is an element of the set A, we would write $x \in A$.

Problem 4. *Do the ordered pairs* $\left\{(x,x^2) : x \in \mathbb{R}\right\}$ *define a function? Explain your answer.*

Problem 5. *Consider the function* f *defined by* $\{(x, f(x))\}$*, where* $f(x) = x^2 - 3x - 4$.

(a.) What is the largest collection of real numbers that could be considered the domain of f*?*

(b.) Find all x *such that* $f(x) = 6$.

(c.) Find all x *such that* $f(x) = -6$.

(d.) *What is the range of* f?

Problem 6. *Suppose the function g is defined by the rule below. What is the largest collection of real numbers that could be considered the domain of g?*

$$g(x) = \frac{x^2+10}{x-5}$$

One way to find out if two functions are the same is to compare their domains (or ranges). If they have different domains, for example, they cannot be the same function.

Problem 7. *Compare the functions defined by the rules* $a(x) = \frac{x^2-9}{x-3}$ *and* $b(x) = x+3$. *How are they similar? Do these rules define the same function? Be able to explain your answer.*

Some functions may have more than number as an input. For example, the collection of ordered *triples* $\{(x,y,f(x,y)\}$ where $f(x,y) = x^2 - y + 6x - 9$ contains the triples

$$(2,0,7) \text{ and } (0,1,-10), \text{ since } f(2,0) = 7 \text{ and } f(0,1) = -10.$$

So in this case the domain would be a collection of ordered pairs in the plane, and the range would be a subset of real numbers.

Problem 8. *Consider the functions defined by the rules below. Can the same values of x and h be used as inputs in both functions? That is, are their domains the same?*

$$A(x,h) = \frac{-3h}{h(x+h-2)(x+2)} \quad \textit{and} \quad D(x,h) = \frac{-3}{(x+h-2)(x+2)}$$

Problem 9. *For the function defined by the rule* $f(x) = x^2 - 2x + 5$,

(a.) *What is* $f(3)$?

(b.) *What is* $f(3+h)$, *where h is some unknown real number?*

(c.) *Calculate and simplify the difference* $f(3+h) - f(3)$.

(d.) *Calculate and simplify the "difference quotient"* $\dfrac{f(3+h)-f(3)}{h}$.

Problem 10. *For the function defined by the rule* $w(x) = \dfrac{2x+3}{x+1}$,

(a.) *What is* $w(1)$?

(b.) *What is* $w(1+h)$, *where* h *is some unknown real number?*

(c.) *Calculate and simplify the difference* $w(1+h) - w(1)$.

(d.) *Calculate and simplify the difference quotient* $\dfrac{w(1+h)-w(1)}{h}$.

Problem 11. *For the function* $g(x) = \sqrt{x-3}$,

(a.) *What is* $g(12)$?

(b.) *What is* $g(12+h)$, *where* h *could be* any *real number?*

(c.) *Calculate and simplify the difference quotient* $\dfrac{g(12+h)-g(12)}{h}$ *by rationalizing the numerator.*

What we've done in the last few problems is compute the **difference quotient** for a few functions at particular points. We'll see this again next chapter, when we'll notice that we're actually computing the slope of a line....

Recall the **slope** of a line in the x-y plane: it is the ratio "rise over run" between any two points on that line. More specifically, if two points are given as (a,b) and (c,d), the slope m of the line (if it exists) between them may be found as:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{d-b}{c-a}$$

Problem 12. *Find the slope of the line between the points:*

(a) $(1,4)$ *and* $(3,-6)$

(b) $(-2,7)$ *and* $(4,7)$

(c) $(5,-1)$ *and* $(5,4)$

(d) *on the graph of the function* $f(x) = x^2+5$ *with* $x=-1$ *and* $x=2$.

Now let's look at a special case of the slope of a line: suppose two points are on the graph of a function f. The slope of the *straight* line between these points will be called the *average rate of change*.

Definition 13. *Suppose f is a function and a & b are two numbers in the domain of f. The* **average rate of change of f between a and b** *is the slope of the straight line between the points $(a, f(a))$ and $(b, f(b))$. That is,*

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Problem 14. *Consider the function $f(x) = x^3 - 3x^2 - 6x + 8$.*

(a.) *Sketch the graph of the function f.*

(b.) *Compute the average rate of change between $x = 2$ and $x = 6$ and sketch the line between these points on the graph of f.*

(c.) *Do the same for $x = 2$ and $x = 3$.*

(d.) *Do the same for $x = 2$ and $x = 2.1$.*

There is one more function which will be used throughout all semesters of calculus: the absolute value function. This function allows us to calculate the distance between two points. The following sequence of questions will help us define this function.

Problem 15. *Answer the following questions in order.*

(a.) *How far apart are 12 and 37 on the number line?*

(b.) *How was this distance found?*

(c.) *Perform exactly the same calculation to compute the distance between 52 and 16.*

(d.) *A negative number was probably obtained as the previous solution. What is done to reconcile this as a* distance*?*

(e.) *How far apart are -13 and 8?*

(f.) *What is the distance between the real numbers x and y?*

We therefore need a function that does the following: if the input is positive, do nothing; if the input is negative, force the output to be positive. (How can this be done? Why would this be useful?)

Definition 16. *For any real number x, define the* **absolute value** *function*

$$abs(x) = |x| = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}$$

The range of this function is the interval $[0, \infty)$.

Problem 17. *Define the function g with domain $\mathbb{R}$ as $g(x) = \sqrt{x^2}$.*

(a.) Show that $g(x) = abs(x)$ for all $x \geq 0$.

(b.) Show that $g(x) = abs(x)$ for all $x < 0$.

Problem 18. *Sketch the graph of the function $abs(x) = |x|$. Find the average rate of change of this function over the interval $[-2, 5]$.*

Problem 19. *Write the rule for a function $f(x,y)$ which will provide the distance between the numbers x and y.*

There are several ways to obtain a new function from two given ones. Some are based on standard arithmetic operations, while another may be less familiar. Suppose f and g are two functions such that $g(x) \neq 0$ for all $x \in \mathbb{R}$. We define the four functions $f \pm g$, $f \cdot g$, and $\frac{f}{g}$ as follows:

$$(f \pm g)(x) = f(x) \pm g(x) \qquad (f \cdot g)(x) = f(x) \cdot g(x) \qquad \frac{f}{g}(x) = \frac{f(x)}{g(x)}$$

A final fifth function only makes sense if every number in the range of the function g is also in the domain of f. In this case, define the **composition** function $f \circ g$ as

$$(f \circ g)(x) = f(g(x)).$$

In the special case in which f and g are two functions which satisfy

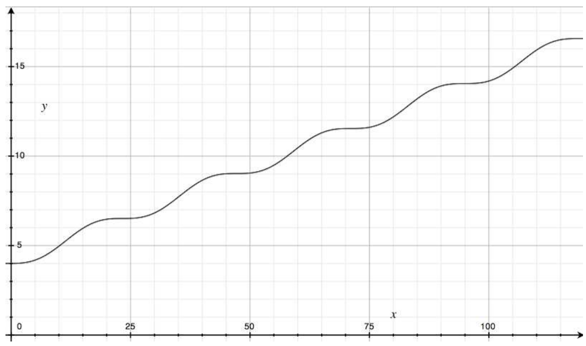
$$(f \circ g)(x) = x \text{ and } (g \circ f)(x) = x,$$

we say f and g are **inverse functions**.

Chapter 1

The Derivative

The function $h(x) = 0.4\left(\sin(\frac{x}{3.8} - 3) + \frac{x}{3.8} - 3\right) + 5.25$ *may be used to describe the height of a 4cm seedling that is planted at noon, where x is the number of hours after planting.*



Is the plant always growing? When does it grow the most? The least? According to the model, how much does it grow each day?

In this chapter we will use the average rate of change of a function f to develop a definition for the *derivative* of f. This derivative will be a function in its own right, and will give us the means to determine many characteristics of the original function f. For example, we will be able to find the largest (or smallest) function values of f, the intervals of the domain on which f is increasing (or decreasing), and many more applications.

First, we will recall the definition of average rate of change from last chapter. Then, we will quickly examine what happens to this rate as the interval gets smaller. Details about "what happens" will be saved for our next chapter on limits.

Definition 20 (A restatement of Definition 13). *Suppose f is a function and a & b are two numbers in the domain of f. The* **average rate of change**

of f **between** a **and** b *is the slope of the straight line between the points* $(a, f(a))$ *and* $(b, f(b))$*:*

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

That is to say, the average rate of change of f *over the interval* $[a,b]$ *is the slope of the* **secant** *line between the points* $(a, f(a))$ *and* $(b, f(b))$ *on the graph of* f.

Problem 21. *Suppose the cost of producing q leather smartphone cases can be found with the function* $C(q) = 100 + 2q - 0.01q^2$.

(a.) What is the average rate of change of this function over the interval $[2,5]$*?*

(b.) What are the units of this measurement?

(c.) Consider the average rate of change of the production costs over the invervals $[2,5]$, $[3,6]$, *and* $[4,7]$. *Does the average rate of change remain constant as the interval changes? Explain.*

Problem 22. *Please refer to the cost function from Problem 21.*

(a.) How much does it cost to produce the 11^{th} *case?*

(b.) Can this be expressed as the average rate of change over an interval?

Problem 23. *Sketch the graph of* $f(x) = -x^3 + 7x^2 - 10x + 2$. *Find the average rate of change of* f *over the interval between* $a = 2$ *and* $b = 3$. *Repeat for* $b = 2.5, 2.1, 1.9, 1.99$

Problem 24. *Compute the average rate of change of* $f(x) = -x^3 + 7x^2 - 10x + 2$ *over the interval* $[2, 2 + \Delta x]$, *where* Δx *is any small nonzero number. Simplify this expression as much as possible. Why is it important that* Δx *be nonzero?*

Problem 25. *In the previous problem, the value of* Δx *could be any small nonzero number. What would happen to the average rates of change if we were to force* Δx *to be closer and closer to zero? Describe what happens to the secant lines as* Δx *gets closer to zero.*

Problem 26. *Compute the average rate of change of $g(x) = \frac{1}{x}$ over the interval $[1, 1 + \Delta x]$. Simplify the expression as much as possible.*

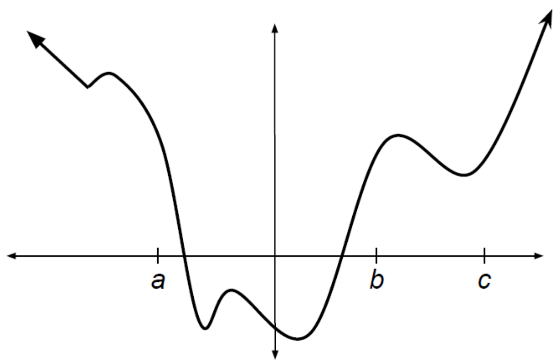
Problem 27. *Compute the average rate of change of $h(x) = \sqrt{x}$ over the interval $[3, 3 + \Delta x]$. Simplify the expression as much as possible.*

Problem 28. *Compute the average rate of change of $m(x) = x^4$ over the interval $[a, a + \Delta x]$. Simplify the expression as much as possible. Give a description of this expression.*

Problem 29. *For the functions g, h, and m above, describe what happens to the expressions for the average rate of change as Δx approaches – but is never exactly – zero. What do these describe?*

Definition 30. *A straight line is said to be* **tangent** *to the graph of f at $x = a$ if that line contains the point $(a, f(a))$ and is that line which is "closest to" the graph of f at that point. We might call the tangent line the "best linear approximation" of the graph of f at $x = a$.*

Problem 31. *Consider the graph of some function below that contains peaks, valleys, and one sharp corner. Sketch the lines tangent to the graph at the points $x = a$, $x = b$ and $x = c$. Are there any values of x at which the tangent lines are horizontal? Are there any values of x at which there is no tangent line?*



Problem 32. *Sketch the graph of the function $f(x) = -x^2 + 4x - 1$. Find the equation of the line that is tangent to this graph at the point $(2, 3)$.*

Problem 33. *If the graph of a function g weren't readily available, how would we go about finding the slope a line tangent to its graph?*

Problem 34. *What is the slope of the line tangent to the graph of* $g(x) = \frac{1}{x}$ *at* $x = 1$*?*

Problem 35. *What is the slope of the line tangent to the graph of* $h(x) = \sqrt{x}$ *at* $x = 3$*?*

Definition 36. *Let* f *be a function with the number* $x = a$ *in its domain. We say the* **derivative of** f **at** $x = a$ **exists** *if there is a line tangent to the graph of* f *at* $x = a$*. In this case, we call the slope of this line* **the derivative of** f **at** $x = a$*.*

Problem 37. *What is the derivative of* $g(x) = \frac{1}{x}$ *at* $x = 1$*?*

Problem 38. *What is the derivative of* $h(x) = \sqrt{x}$ *at* $x = 3$*?*

Problem 39. *What must be true about the function* f *near* $x = a$ *if the derivative of* f *at* a *is* positive*? What if the derivative is* negative *there? In Figure 31 above, find the values of* x *at which the derivative of the function is positive. Negative? What can be said about* f *at points that have derivative equal to zero?*

We are often able to find the slopes of infinitely many tangent lines at the same time, producing a new function. The next two problems will guide the reader through the discovery of this function.

Problem 40. *Consider the function* $E(x) = x^2 + 6x - 5$.

- *(a.) Compute the average rate of change of* E *over the interval* $[a, a + \Delta x]$*, simplifying as much as possible.*
- *(b.) What happens to this expression for the average rate of change as* Δx *approaches zero?*
- *(c.) Use this to find the derivative of* E *at* $x = -1, 0, 1, 2$.

Problem 41. *What is the derivative of the function* $B(x) = \frac{3}{x+2}$ *at* $x = a$*? Use this to find the slope of the lines tangent to the graph of* B *at* $x = -1, 0, 1, 2$.

Now we're ready for a rigorous definition of the derivative of a function. We'll see an even more mathematical (and hence more complete and concise) definition in Chapter 3.

Definition 42. *Let f be any function. The function f' that assigns to a value x the slope of the line tangent to the graph of f is called the* **derivative function of** f*. It is found by first finding an expression for the average rate of change of f, $\frac{\Delta f}{\Delta x}$, over the interval $[x, x+\Delta x]$, then forcing Δx to approach zero.*

Problem 43. *If $f(x) = x^2 - 5x + 7$, calculate the derivative of f. Give it the name f'. Compare the domains of the functions f and f'.*

Problem 44. *If $h(x) = \sqrt{x}$, then calculate $h'(x)$. Compare the domains of the functions h and h'.*

Problem 45. *Recall Definition 16. If $a(x) = |x|$, then calculate $a'(x)$.*

Problem 46. *For the cost function of Problem 21, compute the derivative function $C'(x)$. In this context, we call this function the* **marginal cost** *function. What do these function values describe? What are the units for the function values $C'(x)$?*

Chapter 2

Limits

> *The most difficult subjects can be explained to the most slow-witted man if he has not formed any idea of them already; but the simplest thing cannot be made clear to the most intelligent man if he is firmly persuaded that he knows already, without a shadow of doubt, what is laid before him.*
>
> – Tolstoy, 1897

In the last chapter, we developed the derivative function as the result of a *process*. This process can be summarized as: letting the width of an interval get smaller (towards zero), while keeping track of what happens to the average rate of change over this interval. This is an example of the procedure called "taking a limit." We defined the derivative as a particular limit of a particular function.

Before learning about the properties and uses of the derivative, we will look more closely at the concept of limits. Not only will limits allow us to give a precise definition (and interpretation) of the derivative of a function, it will serve as the backbone of calculus. Every concept in all of calculus (usually three semesters!) is based on limits, and no real learning of calculus can be had without a firm understanding of limits.

Problem 47. *Consider the function $r(x) = \frac{1}{x}$, for $x > 0$. As the values of x get closer to zero, is there a number to which the corresponding values of $r(x)$ get closer? That is, as x approaches zero (from the right), is there a number that the values of $r(x)$ approach?*

Problem 48. *Consider the function $s(x) = \frac{\sin x}{x}$, for $x \neq 0$. As the values of x approach 0, is there a number that the corresponding values of $s(x)$ get closer to? How do you know?*

Problem 49. *Consider the function $t(x) = \frac{x^2+7x+28}{x-5}$, for $x \neq 5$. As the values of x get closer to 5, is there a number that the corresponding values of $t(x)$ get closer to? That is, as x approaches 5, do the values of $t(x)$ approach a number?*

Problem 50. *Consider the function $f(x) = \frac{x^2-2x-3}{x^2-3x}$, for $x \neq 3$. As the values of x get closer to 3, is there a number that the corresponding values of $t(x)$ get closer to? That is, as x approaches 3, do the values of $f(x)$ approach a number?*

Problem 51. *Consider the functions $f(x) = \frac{x^2-2x-3}{x^2-3x}$ and $g(x) = \frac{x+1}{x}$.*

(a.) Are these the same functions? If not, how do they differ?

(b.) For what values of x will the function values $f(x)$ and $g(x)$ be equal?

(c.) How do their graphs differ?

Problem 52. *What happens to the values of $f(x) = \frac{x^2-2x-3}{x^2-3x}$ as $x \to 0$?*

Problem 53. *What happens to the values of $h(x) = \frac{x^2+2x-15}{x^2-9}$ as $x \to 3$?*

Problem 54. *What happens to the values of $h(x) = \frac{x^2+2x-15}{x^2-9}$ as $x \to -3$?*

Definition 55 (Informal definition of existence of a limit). *We say the* **limit of f as x approaches** *a is equal to L if, as x gets closer to a, the corresponding values of $f(x)$ get closer to L. We write $\lim_{x \to a} [f(x)] = L$.*

This definition says that if the inputs of f get as close as they want to a, the outputs are *forced* to get close to a number L. Let's be more careful when we use the phrase "gets closer to...."

Problem 56. *Consider the function $F(x) = \frac{2x^2-50}{x-5}$. Give an educated guess for $\lim_{x \to 5} [F(x)]$. If we require the values of $F(x)$ to be within 0.1 of this guess, how close must our values of x be to 5? [Drawing a picture may be helpful.]*

Problem 57. *Consider the function $F(x) = \frac{2x^2-50}{x-5}$. If we require the values of $F(x)$ to be within 0.01 of this guess, how close must our values of x be to 5?*

Problem 58. *Consider the function $F(x) = \frac{2x^2-50}{x-5}$. If we require the values of $F(x)$ to be within some tiny positive number ε of this guess, how close must our values of x be to 5?*

Problem 59. *Consider the function $G(x) = \frac{7x^2+14x-105}{4x-12}$. Give an educated guess for $\lim\limits_{x\to 3}[G(x)]$. If we require the values of $G(x)$ to be within some tiny positive number ε of this guess, how close must our values of x be to 3?*

Here we give a *formal* definition of the existence of a limit. We'll use the absolute value function to help us compute distance.

Definition 60 (Formal definition of the existence of a limit.). *Let $\varepsilon > 0$ be any small number. We say that* **the limit of** $f(x)$ **as** x **approaches** a **is equal to** L *(denoted $\lim\limits_{x\to a}[f(x)] = L$) if we can find a positive number δ so that for any values of x which satisfy*

$$0 < |x-a| < \delta,$$

it must be true that the function values $f(x)$ satisfy

$$|f(x) - L| < \varepsilon.$$

That is, in order for the limit of $f(x)$ to be equal to the number L as x approaches a, the following must be true: no matter how close we require our function values $f(x)$ be to the number L (here, close is defined by the value of ε), there is some interval around a (of width δ) so that any value of x inside this interval will have function value $f(x)$ that close to L.

This is a complicated definition, but it allows us to provide a quantifiable definition of the phrase "close to." Let's use this rigorous definition to *prove* something we earlier suspected was true.

Problem 61. *Fill in the blanks in the following proof that*

$$\lim_{x\to 5}[F(x)] = \lim_{x\to 5}\left[\frac{2x^2-50}{x-5}\right] = 20$$

Let $\varepsilon > 0$ be any small number.

Set δ equal to the positive number ________.

For those x which satisfy $0 < |x - \_\_\_\_\_\_\_\_| < \delta$, we have

$$\begin{aligned}
|F(x) - __________| &= \left|\frac{2x^2-50}{x-5} - _____\right| \\
&= \left|\frac{2(________)(________)}{x-5} - _____\right| \\
&= |2(________) - 20| \\
&= |2x - ________| \\
&= 2\cdot|________| \\
&< 2\cdot\delta \\
&= 2\cdot________ \\
&= \varepsilon
\end{aligned}$$

So, as long as x is within $\delta =$ ________ of 5, the values of $F(x)$ will be within ε of 20, for ANY positive value of ε.

Problem 62. *Provide an educated guess for the value of $\lim_{x\to -7}[E(x)]$, where $E(x) = \frac{x^2+11x+28}{2x+14}$.*

Problem 63. *Fill in the blanks in the following proof that your educated guess in Problem 62 is correct.*

Let $\varepsilon > 0$ *be any small number.*

Set δ *equal to the positive number* _______.

For those x which satisfy $0 < |x - \_\_\_\_\_\_\_| < \delta$*, we have*

$$\begin{aligned} |E(x) - \_\_\_\_\_\_\_\_ | &= \left| \frac{(\quad\quad)(\quad\quad)}{2(\quad\quad)} - \_\_\_\_ \right| \\ &= \left| \frac{(\_\_\_\_\_\_)}{2} - \_\_\_\_ \right| \\ &= \left| \tfrac{1}{2}(\_\_\_\_\_\_) + \tfrac{3}{2} \right| \\ &= \tfrac{1}{2} \cdot |\_\_\_\_\_\_| \\ &< \tfrac{1}{2} \cdot \delta \\ &= \tfrac{1}{2} \cdot \_\_\_\_\_\_ \\ &= \varepsilon \end{aligned}$$

So, as long as x is within $\delta =$ _______ *of* -7*, the values of* $E(x)$ *will be within* ε *of* _______ *, for ANY positive value of* ε.

Problem 64. *Give an educated guess for* $\lim_{x \to -4} \left[\dfrac{2x^2 + 5x - 12}{5x + 20} \right]$. *Prove that your guess is the correct one.*

There are many properties of limits that will help us both now with the computation of limits as well as later in the development of the derivative. The first one states that the limit of the sum (or difference) of two functions is equal to the sum (or difference) of the limits of these functions, <u>as long as these limits exist</u>! We've actually been using this property for some time without actually seeing it stated:

$$\text{If } \lim_{x \to a} [f(x)] = L \text{ and } \lim_{x \to a} [g(x)] = K, \text{ then } \lim_{x \to a} [f(x) \pm g(x)] = L \pm K.$$

The same is true for products, quotients, and scalar multiples. We will summarize this in the statement of a theorem.

Theorem 65. *Suppose that*

$$\lim_{x\to a}[f(x)] = L \text{ and } \lim_{x\to a}[g(x)] = K. \text{ Then we know}$$

(a.) $\lim_{x\to a}[f(x)\pm g(x)] = L\pm K$.

(b.) $\lim_{x\to a}[f(x)\cdot g(x)] = L\cdot K$.

(c.) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$*, as long as* $K\neq 0$.

(d.) For any real number c*,* $\lim_{x\to a}[c\cdot f(x)] = c\cdot L$.

Problem 66. *Compute the following limit by rationalizing the numerator:*

$$\lim_{x\to 4}\left[\frac{\sqrt{x}-2}{x-4}\right]$$

Problem 67. *Compute each of the following limits:*

$$\lim_{x\to -1}\left[(4x+5)^{12}\right] \qquad \lim_{x\to 9}\left[\frac{\sqrt{x}-3}{x-9}\right] \qquad \lim_{x\to 2}\left[\frac{x^3-8}{x^4-16}\right]$$

There are several strategies we can use to compute the limit of $f(x)$ as $x\to a$. If the value of a is not in the domain of the function, the strategy used most often up to this point has been to try and "cancel" that part of the function which prevents a from being in the domain. Another strategy is outlined below. Commonly known as the Sandwich or Squeeze Theorem, it allows us to find the limit of a function that finds itself bounded between two other functions. In other parts of the world it is also known as the "two policemen and a drunk" theorem.

Theorem 68 (The Sandwich Theorem). *Suppose f, g, and h are functions which satisfy*

(i.) $f(x) \leq g(x) \leq h(x)$ *for all values of x that are "near" a, and*

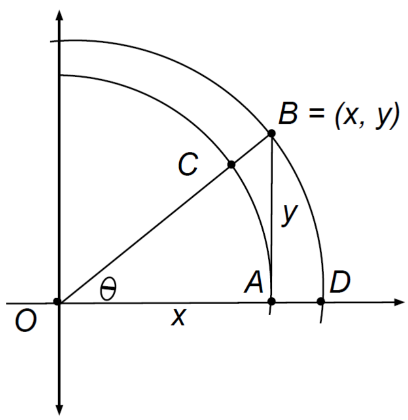
(ii.) $\lim_{x\to a}[f(x)] = L = \lim_{x\to a}[h(x)]$.

Then $\lim_{x\to a}[g(x)] = L$ *as well.*

Before we make use of the Sandwich Theorem, it may be useful to recall some facts from a trigonometry course regarding a sector of a circle.

Problem 69. *Suppose that in a circle of radius r, an angle of size θ radians creates a sector. What is the area of this sector?*

Let's use the Sandwich Theorem to find a very important limit involving trigonometric functions. Consider the construction below, which was made based on some angle θ between 0 and $\pi/2$ radians. The larger arc has been drawn to have radius equal to 1. (That is, $\overline{OB} = 1$.)



Problem 70. *Using the construction above, answer the following questions.*

(a.) Use the diagram above to rank the sector OBD, the sector OCA, and the triangle OBA in order from smallest to largest area

(b.) What is the area of the triangle OBA?

(c.) What is the area of the sectors OBD and OCA?

Problem 71. *Use the Sandwich Theorem to compute* $\lim_{\theta\to 0}\left[\frac{\sin\theta}{\theta}\right]$

Problem 72. *Compute* $\lim\limits_{\theta\to 0}\left[\dfrac{\sin 4\theta}{4\theta}\right]$

Problem 73. *Compute* $\lim\limits_{\theta\to 0}\left[\dfrac{\sin 4\theta}{7\theta}\right]$

Problem 74. *Use a relevant trigonometric identity to compute the limit* $\lim\limits_{\theta\to 0}\left[\dfrac{\cos\theta - 1}{\theta}\right]$.

Problem 75. *Compute* $\lim\limits_{x\to 0}\left[x^2\cos(\tfrac{1}{x})\right]$

Chapter 3

Uses of the Derivative

> *The calculus is the greatest aid we have to the appreciation of physical truth in the broadest sense of the word.*
>
> – W. F. Osgood

In Chapter 2 we defined the derivative f' as that function whose value $f'(x)$ at x is the slope of the line tangent to the graph of f at the point $(x, f(x))$. Now that we know about the existence of limits, we are able to develop a more concise definition of the derivative function. But this level of rigor comes at the cost of being less intuitive.

Definition 76. *For a function f, the* **derivative** f' **of** f **at** x *is the function defined by the following limit, provided that it exists:*

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right].$$

$$\text{That is, } f' = \lim_{\Delta x \to 0} \left[\frac{\Delta f}{\Delta x} \right]. \quad \text{We often write } f' = \frac{df}{dx}.$$

If this limit does not exist, we say the derivative of f does not exist at x.

Definition 77. *If the function f has a derivative at x, we say the function is* **differentiable at** x. *If a function is differentiable at every point in a particular set (e.g., its domain), we say the function is differentiable on that set.*

Problem 78. *Suppose f is the linear function $f(x) = mx + b$ for some constants m and b. Use the definition above to calculate the derivative f'.*

Definition 79. *In the special case in which the linear function $f(x) = mx + b$ has $m = 1$ and $b = 0$, we have $f(x) = x$. We call this particular function* **the**

identity function.

Problem 80. *Suppose g is the quadratic function $g(x) = ax^2 + bx + c$ for some constants a, b and c. Compute the derivative g'.*

Problem 81. *Compute the derivative of each of the following functions.*

$$f(x) = 7x - 4 \qquad g(x) = 5 - 10x \qquad h(x) = 4x^2 - 7x + 8$$
$$k(x) = 1 - 10x^2 \qquad \ell(x) = (1 - 2x)^2$$

Since the derivative is simply the limit of a special function, we may use the properties of limits to develop similar properties for derivatives. The next theorem is simply a restatement of the properties from the previous chapter in the context of derivative functions.

Theorem 82. *Suppose that f and g are functions which have a derivative at x. Then for any constant k,*

(a.) $(f \pm g)'(x) = f'(x) \pm g'(x)$

(b.) $(k \cdot f)'(x) = k \cdot f'(x)$

Problem 83. *Use Definition 76 and properties of limits to prove the previous theorem.*

Problem 84. *Calculate the derivative of the functions $a(x) = 4x, b(x) = 3x$, and $c(x) = 12x^2$.*

Problem 85. *Is it true that $(f \cdot g)' = f' \cdot g'$ for any differentiable functions f and g?*

Problem 86. *Suppose f and g are differentiable functions. Use Definition 76 to compute the derivative of the function $f \cdot g$ by adding and subtracting the quantity $f(x) \cdot g(x+h)$ at the appropriate time.*

Theorem 87 (The Product Rule). *For any functions f and g which have a derivative x, we have $(f \cdot g)'(x) = \ldots$*

Problem 88. *Use the Product Rule to compute the derivative of* $f(x) = x^3$.

Problem 89. *Compute the derivative of the function* $f(x) = x^4$.

It may be useful now to recall what we know about positive powers of binomials: $(x+y)^n$, where $n \geq 1$. The coefficients for the terms in this expansion correspond to the n^{th} row of what is commonly known as Pascal's triangle (if we start counting rows at 0). The coefficients in front of each term of $(x+y)^2$ we know are

$$1 \quad 2 \quad 1,$$

while the coefficients in front of each term of $(x+y)^5$ will be

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1.$$

But we can do even better than this. Each of these coefficients may be written as a *combination*: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where $k = 0, 1, \ldots n$. So

$$\begin{aligned}(x+y)^5 &= \tbinom{5}{0} \cdot x^5 + \tbinom{5}{1} \cdot x^4 y + \tbinom{5}{2} \cdot x^3 y^2 + \tbinom{5}{3} \cdot x^2 y^3 + \tbinom{5}{4} \cdot x y^4 + \tbinom{5}{5} \cdot y^5 \\ &= x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5xy^4 + y^5\end{aligned}$$

That is, the k^{th} coefficient in the expansion of $(x+y)^n$ is equal to $\binom{n}{k}$ for $k = 1, 2, \ldots n$. Use this to solve the next problem.

Problem 90. *Let n be a positive integer. Compute* f' *if* $f(x) = x^n$.

Theorem 91 (The Power Rule for positive integer powers). *Let n be a positive integer. The derivative of the monomial* $f(x) = x^n$ *is ....*

Problem 92. *Calculate the derivative of the function* $h(x) = (5x^3 + 2x^2 - 10x + 5) \cdot (x^4 + 6x^2 - 1)$.

Problem 93. *Calculate the derivative of the function* $p(x) = (3x^2 + x - 1)^3$.

We now know how to differentiate products of functions. Before we see the technique for finding the derivative of the quotient of two functions (or other combinations of functions), we pause to learn the first use of the derivative.

Problem 94. *What must be true about a function* f *near* $x = a$ *if* $f'(a) > 0$*? If* $f'(a) < 0$*? What about* $f'(a) = 0$*?*

Definition 95. *The point* $(c, f(c))$ *is a* **local maximum** *for the function* f *if* $f(c) > f(x)$ *for all values of* x *that are "near"* c*.*

Definition 96. *The point* $(c, f(c))$ *is a* **local minimum** *for the function* f *if* $f(c) < f(x)$ *for all values of* x *that are "near"* c*.*

Problem 97. *Is it possible for a function to have a local max. or min. at* $x = c$ *but not have a derivative at* $x = c$*?*

Problem 98. *If* $x = c$ *is a local maximum (or minimum) for* f *and* $f'(c)$ *exists, then what must be true about the value of* $f'(c)$*?*

Problem 99. *State the results of the previous problem as a theorem. Give this theorem a name.*

Problem 100. *Is the converse of the theorem from Problem 99 true? If not, provide a counterexample.*

Definition 101. *Suppose* c *is in the domain of the function* f*. We say* c *is a* **critical point** *for* f *if either* $f'(c) = 0$ *or* $f'(c)$ *does not exist.*

Problem 102. *Sketch the graph of a function with four critical points, one local maximum, two local minima, and one point with no derivative.*

Problem 103. *Sketch the graph of a function with five critical points at* $x = -2, 0, 2, 4, 7$*, one local maximum at* $x = 7$*, two local minima at* $x = \pm 2$*, and one point with no derivative.*

Problem 104. *Determine all critical points for the function* $f(x) = \frac{1}{5}x^5 - \frac{3}{4}x^4 - \frac{10}{3}x^3$*.*

Problem 105. *Determine all critical points for the function* $f(x) = \frac{1}{5}x^5 - \frac{3}{4}x^4 + \frac{10}{3}x^3$*.*

Problem 106. *Determine all critical points for the function*
$g(x) = \begin{cases} x^2 & \text{for } x \leq 1 \\ 3-2x & \text{for } x \geq 1 \end{cases}.$

Problem 107. *Determine all critical points for the function f if $f'(x) = \dfrac{x^2-x-6}{x^2+x}$.*

Problem 108. *Suppose the function g is differentiable for all values of x. If g has exactly two critical points a and b, then*

(a.) *what can be said about the values of $g'(x)$ for $x \in (a,b)$?*

(b.) *what can be said about the function g for $x \in (a,b)$?*

Problem 109. *If c is a critical point for a function f, how could it be determined if c is a local maximum or local minimum?*

Problem 110. *Develop a test for finding all local maxima and minima for a differentiable function. Give this test a name.*

Problem 111. *Use the test from Problem 110 to find all extrema on the function $g(x) = x^3 + x^2 - 8x + 5$.*

Problem 112. *Use the test from Problem 110 to find all extrema on the function f from Problem 107.*

Problem 113. *Geotech Industries owns an oil rig 12 miles off the shore of Galveston. This rig needs to be connected to the closest refinery, 20 miles down the coast from the rig. If underwater pipe costs \$500,000 per mile and above-ground pipe costs \$300,000 per mile, the company would like to know which combination of the two will cost Geotech the least amount of money. Find a function that may be minimized to find this cost.*

Problem 114. *What would have to be true about the graph of a function f if the derivative of f was increasing near x?*

Problem 115. *If f had a local maximum at the critical point $x = c$, then what can be said about the value of $f''(c)$? And if $x = c$ were a local minimum?*

Problem 116. *Develop another (quicker?) test for finding all local maxima and minima for a function. Give this test a name.*

Problem 117. *Use the test from Problem 116 to find all extrema on the function* $f(x) = -2x^3 + 6x^2 - 3$.

Problem 118. *Use the test from Problem 116 to find all extrema on the function* $g(x) = 4x^3 - x^4$.

Problem 119. *Suppose we know the derivative of a function* h *is* $h'(x) = x(x-3)^2$. *Sketch the general shape of the graph of* h.

Chapter 4

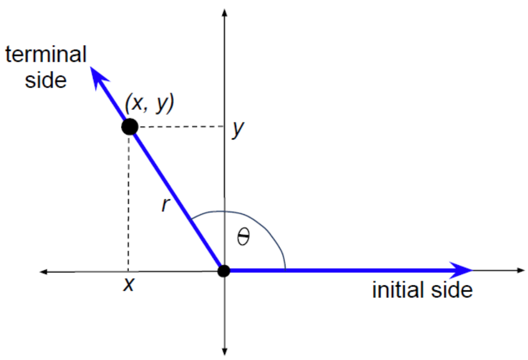
Trigonometric Functions

> *A particular species of hibiscus plant grows at a rate that can be calculated using the model $h = 0.2t + 0.03\sin(2\pi t)$, where t is measured in days (and $t = 0$ corresponds to midnight on the first day). Interpret the validity of this model. When is this plant growing the fastest? The slowest?*

In the previous chapters, we learned how to find the derivative of any polynomial function, as well as products of polynomials. In this chapter, not only will we learn how to differentiate *quotients* of functions, we'll also discover the derivatives of trigonometric functions.

First, a review of the definitions of the basic trigonometric functions. The "input" for these functions will be real numbers denoted θ.

For any real number θ, form an angle of θ *radians* whose initial side is the positive x-axis. Choose **any** point with coordinates (x, y) on the terminal side of this angle. Let r be the distance from this point to the origin $(0, 0)$.



Definition 120. *Using the above construction – for any real number θ – define the* **sine and cosine functions** *by the following ratios:*

$$\sin(\theta) = \frac{y}{r} \qquad \cos(\theta) = \frac{x}{r}$$

Problem 121. *Use the preceding definition to compute* **the exact values** *of each of the following:*

$$\sin(0) \qquad \cos(0) \qquad \sin(\pi) \qquad \cos(\pi)$$

$$\sin(\pi/2) \qquad \cos(\pi/4) \qquad \sin(\pi/4) \qquad \cos(3\pi/2)$$

Problem 122. *Does it matter if the point (x,y) or another point (u,v) on the terminal side of the angle is chosen when defining these functions? What principle of geometry can be used to justify your answer?*

Problem 123. *Use the definitions of sine and cosine to compute and simplify the quantity $\sin^2\theta+\cos^2\theta$ for <u>any</u> angle θ.*

In the next theorem, we are reminded of several identities involving the sine and cosine functions. We will use each of these several times, so learn them well. One of them may be proven from the construction used to define the functions.

Theorem 124. *For any angles A and B, the following are all satisfied:*

$$\sin(A\pm B)=\sin(A)\cos(B)\pm\sin(B)\cos(A)$$

$$\cos(A\pm B)=\cos(A)\cos(B)\mp\sin(A)\sin(B)$$

Problem 125. *Use the definition of the derivative (and maybe a trig. identity?) to find the derivative of $\sin x$.*

Problem 126. *Use the definition of the derivative to calculate the derivative of $\cos x$.*

Problem 127. *If $f(x)=x\cdot\cos x$, then calculate f'.*

Problem 128. *If $f(x)=\sin x\cdot\cos x$, then calculate f'.*

Problem 129. *If* $g(x) = \cos x \cdot (x^2 - 6x + 8)$*, compute the derivative of* g*.*

Problem 130. *Consider the function* $f(x) = \sin x$ *on the interval* $[-\pi, 3\pi]$*. Find all local maxima and minima on this interval. Find all values of* x *such that the tangent line at* x *has slope 1.*

Definition 131. *We define the* **tangent, cotangent, secant, and cosecant functions** *as follows:*

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

Note that the domain of each of these function consists of those values of x *for which the denominators are nonzero.*

In order to calculate the derivative of each of these new functions, we will need to know how to find the derivative of the *quotient* of two functions. But first, we'll examine a very particular quotient.

Problem 132. *Suppose* $g(x) \neq 0$*. Use Definition 76 to compute the derivative of the quotient* $\dfrac{1}{g(x)}$*.*

Problem 133. *If* $x > 0$ *and* $f(x) = \frac{1}{x^4}$*, then calculate* f'*.*

Problem 134. *Suppose* $g(x) \neq 0$*. Use a strategy similar to what was used when proving the Product Rule to compute the derivative of the quotient* $\dfrac{f(x)}{g(x)}$*.*

Theorem 135 (The Quotient Rule). *For any functions* f *and* g*, the derivative of the quotient* $\dfrac{f(x)}{g(x)}$ *is ....*

Problem 136. *Differentiate the functions*

$$r(x) = \frac{x^4}{x^2+1}, \qquad s(x) = \frac{x^3+6x^2-2x}{x^2+1}, \qquad t(x) = \frac{x^2}{\sin x}$$

Problem 137. *Compute (and simplify) the derivatives of the functions defined in Definition 131.*

Problem 138. *Differentiate the functions*

$$a(x) = \frac{\sin x}{\tan x}, \qquad b(x) = \sin^2 x, \qquad c(x) = x \tan^2 x$$

We may now use the Quotient Rule to extend our Power Rule (Theorem 140) to include negative exponents.

Problem 139. *Let n be a positive integer. Use the Quotient Rule to compute (and simplify) the derivative of the function* $f(x) = x^{-n}$.

Theorem 140 (The Power Rule for integer powers). *Let n be any integer. The derivative of the monomial* $f(x) = x^n$ *is ....*

Problem 141. *Compute the derivative of the function* $f(x) = x^4 - 2x^3 + 8x + \frac{1}{x} - \frac{3}{x^2} + \frac{7}{x^5}$.

Problem 142. *Compute the derivative of the function* $g(x) = \frac{x^7+1}{x^5}$ *without explicitly using the Quotient Rule.*

Problem 143. *Compute the derivative of the functions* $a(\theta) = \sec\theta\tan\theta$ *and* $b(\theta) = \sin\theta\csc\theta$.

Chapter 5

Exponential and Logarithmic Functions

> *Who has not been amazed to learn that the function $y = e^x$, like a phoenix rising from its own ashes, is its own derivative?*
>
> – Francois le Lionnaise

In this chapter, we expand our circle of functions that we are able differentiate to include two new classes of functions: exponential functions and logarithmic functions. These functions are perhaps most valuable to biologists (e.g., in population models) and to chemists (e.g., the decay of radioactive elements). In addition, we see how to find the derivative of the composition of two (or more) functions.

Problem 144. *On the same set of axes, sketch the graphs of the functions $g(x) = 2^x$ and $h(x) = 3^x$.*

Problem 145. *Use a calculator to estimate the value of $\lim\limits_{h\to 0} \dfrac{2^h - 1}{h}$ to three decimal places. Use this estimate to compute the derivative of $g(x) = 2^x$. What is the slope of the tangent line at $x = 0$?*

Problem 146. *Use a calculator to estimate the value of $\lim\limits_{h\to 0} \dfrac{3^h - 1}{h}$ to three decimal places. Use this estimate to compute the derivative of $h(x) = 3^x$. What is the slope of the tangent line at $x = 0$?*

Definition 147. *To simplify later computations, we will introduce some notation. We define* **the number** w_b *to be*

$$w_b = \lim_{h\to 0} \frac{b^h - 1}{h}.$$

Definition 148. *We define* **the number** *e to be that number which causes* $w_e = \lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

Problem 149. *Use the previous definition to estimate the value of the number e to three decimal places.*

Problem 150. *Sketch the graph of the function $f(x) = e^x$. Compute the derivative f'.*

Problem 151. *Compute the derivative y' for each of the following:*

$$y = 10^x, \qquad y = x^{10}, \qquad y = 10^{-x}$$

Problem 152. *Compute the derivative y' for each of the following:*

$$y = xe^x, \qquad y = \frac{x}{e^x}, \qquad y = x^2e^x, \qquad y = x^2 2^x$$

Problem 153. *Compute the derivative y' for each of the following:*

$$y = e^x \sin x, \qquad y = \frac{\tan x}{e^x}, \qquad y = e^{2x}, \qquad y = e^{3x}$$

For the remainder of this chapter, it may be useful to recall the definition of *the identity function* from the beginning of Chapter 3.

Problem 154. *Let* $f(x) = 7x + 3$.

(a.) Describe in words the rule of this function.

(b.) Is there a sequence of mathematical steps that would "undo" what the function f does to x? If so, write these steps as a function named $\overline{f}$.

(c.) Define a new function with the rule "do f, followed by $\overline{f}$." Can this rule be simplified?

(d.) *Define a new function with the rule "do $\overline{f}$, followed by f." Can this rule be simplified?*

Problem 155. *If possible, repeat each part of the previous problem with the function $g(x) = x^2$. If not, explain the reason it is not possible.*

Definition 156. *Suppose there are two functions f and $\overline{f}$ such that f followed by $\overline{f}$ and $\overline{f}$ followed by f are both the same as the identity function $i(x) = x$. Then we say f and $\overline{f}$ are* **inverse functions**.

Problem 157. *Which of the following functions have an inverse?*

$$x^2 \qquad x^3 \qquad \sin x \qquad e^x \qquad x^3 + x^2 - 2x \qquad \tan x \qquad 10^x$$

We have just discovered a theorem about the existence of inverse functions:

Theorem 158. *The function f has an inverse precisely when ....*

Since we know any exponential function of the form $f(x) = b^x$ for $b \neq 0$ is "invertible", we can give its inverse function a special name:

Definition 159. *The inverse of the exponential function $f(x) = b^x$ is called the* **logarithmic function in base** b *and is written $\overline{f}(x) = \log_b(x)$.*

Problem 160. *Complete the following*

(a.) $4^{\log_4(6)} =$? $\qquad$ $4^{\log_4(x)} =$?

(b.) $\log_7(7^9) =$? $\qquad$ $\log_7(7^x) =$?

(c.) $5^3 =$? $\qquad$ $\log_5(125) =$?

(d.) $b^0 =$? $\qquad$ $\log_b(1) =$?

(e.) $b^1 =$? $\qquad$ $\log_b(b) =$?

(f.) $\ln e =$? $\qquad$ $\ln 1 =$?

The following should be a review of some of the concepts involving logarithmic functions seen in a precalculus course.

Theorem 161. *The following three properties are true for all logarithms, regardless of the base b.*

(a.) *For any positive x and y, it is true that* $\log_b(xy) = \log_b(x) + \log_b(y)$.

(b.) *For any positive x and y, it is true that* $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$.

(c.) *For any positive x and any* $n \in \mathbb{R}$, *it is true that* $\log_b(x^n) = n \cdot \log_b(x)$.

Now that we know (and have given names to) the inverse of each exponential function, let's find their derivative. We'll actually use a tool that will help us find the derivative of *any* inverse function, in terms of the derivative of the original function. This tool tells us how to find the derivative of the *composition* of two functions. The proof is beyond the scope of this class, so it will have to wait until an elementary analysis course.

Theorem 162 (The Chain Rule). *Given two functions* f *and* u, *consider their composition* $f \circ u$ *defined by the rule*

$$(f \circ u)(x) = f(u(x)).$$

The derivative of this function is defined by the product

$$(f \circ u)'(x) = f'(u(x)) \cdot u'(x).$$

That is, the derivative of the composition of two functions is equal to the derivative of the "outside" function evaluated at the "inside" function, times the derivative of the "inside" function.

Problem 163. *Compute the derivative* y' *for each of the following:*

(a.) $y = \sin(x^2)$

(b.) $y = e^{5x}$

(c.) $y = (4x^2 - 6x + 7)^6$

(d.) $y = \sqrt{x^2 + 1}$

(e.) $y = \sec 5^x$

(f.) $y = 5^{\sec x}$

Now we're ready to use the Chain Rule to develop a procedure to find the derivative of the *inverse* of a function. We'll use this procedure to compute the derivative of all logarithm functions.

Problem 164. *If $\overline{f}$ is the inverse of the function f, we know that $f(\overline{f}(x)) = x$ for every x (in the domain of $\overline{f}$, of course). Differentiate both sides of this equation and solve for the derivative of $\overline{f}$.*

Problem 165. *Use Problem 164 for the special case $f(x) = b^x$ to find the derivative of $\overline{f}(x) = \log_b(x)$.*

Problem 166. *Find the derivative y' for the following functions:*

$$y = \log_4(x) \qquad y = \log_{10}(x) \qquad y = \log_{10}(x^2+4)$$
$$y = \ln(x) \qquad y = \cos x \cdot \log_2(x)$$

Often we encounter equations involving two variables x and y in which (1.) we know *y depends on x* somehow, but (2.) we can't solve the equation for y to see this dependence explicitly. If we still want to see how the variable x affects the variable y, however, we need to calculate the derivative of y. We can do this using an application of the Chain Rule, a process called *implicit differentiation*.

Problem 167. *Consider the equation $y = x^2 \cdot \sin y$.*

(a.) Treat y as simply some unknown function $y(x)$ and use the Chain Rule to differentiate both sides of the equation.

(b.) Solve for y'.

Problem 168. *Use implicit differentiation to find y' if $xy^2 + 3xy + 5y = e^{7y}$.*

Problem 169. *Use implicit differentiation to find y' if $y = x^{4/3}$.*

We may now use the Chain Rule to extend our Power Rule (Theorem 140) to include rational exponents. Provide a proof of the following theorem.

Theorem 170 (The Power Rule for rational powers). *Suppose $\frac{p}{q}$ is a rational number. The derivative of the function $f(x) = x^{p/q}$ is ....*

Problem 171. *Use implicit differentiation to find y' if $y = 10^x$, by first taking the natural log of both sides of the given equation.*

Problem 172. *Use the previous problem to compute a better definition of the number w_{10}.*

Problem 173. *Use implicit differentiation to find y' if $y = b^x$, by first taking the natural log of both sides of the given equation.*

Problem 174. *Use the previous problem to compute a better definition of the number w_b.*

The procedure used in the last few problems is called *logarithmic differentiation*, in which the natural logarithm (and its properties) is applied to both sides of an equation, and implicit differentiation is used to compute the derivative.

Theorem 175 (The Power Rule for real powers). *Suppose r is any real number. The derivative of the function $f(x) = x^r$ is ....*

Problem 176. *Compute the derivative of the function $f(x) = x^x$.*

Chapter 6

More Applications of the Derivative

> *Calculus is the most powerful weapon of thought yet devised by the wit of man.* – W.B. Smith

Now that we can compute as many derivatives as we please, we're able to put them to use in some more applications.

Problem 177. *If r is a function that gives the distance a particle has moved (in meters) after t seconds, what does r measure? What are the units of the function values $r'(t)$? What about the units for $r''(t)$? What does this second derivative measure?*

Problem 178. *A boy on the top of a ladder throws a ball upwards. The function $r(t) = 100 + 13t - 16t^2$ describes the height of the ball in feet t seconds after the boy throws the ball.*

(a.) How tall is the ladder? How irresponsible are the parents?

(b.) How fast is the ball moving the instant it leaves his hand?

(c.) How long does it take for the ball to reach its highest point? How high does it go?

(d.) How long until the ball reaches the ground?

When a quantity x changes over time, we may think of x as a function of a time parameter t. That is, we may write x as a function of t: $x(t)$. The derivative of x with respect to t, or $x'(t)$, will therefore measure the **rate of change of x with respect to t**. That is, $x'(t)$ is a function that tells us *how fast* the quantity x is changing.

Problem 179. *Write the area of a circle in terms of its radius. Assuming the radius changes over time (i.e. is a function of t), write an equation that relates the rate at which the area changes to the rate at which the radius changes.*

Problem 180. *A prisoner escapes on foot at the center of a large metropolitan area. The police department wants to specify a circle (which will of course expand over time) in which to search for the missing fugitive. If the fugitive can be assumed to move at a rate no more than 6 miles per hour, how fast is the area of this "search circle" expanding two hours after escape?*

Problem 181. *A vertical cylindrical tank allows liquid to be drained at the rate of 200 L/min. How fast will the level of the fluid drop? (The answer will depend on the radius of the cylinder.)*

Problem 182. *A spherical balloon is inflated at a rate of 200 ft^3/min. How fast is its radius changing at the instant its radius is 2 feet? At the instant it is 5 feet?*

Problem 183. *A tanker 50 miles from the shore is leaking oil at the rate of 2000 m^3/min. Assuming the oil disperses in a circular pattern 1 meter deep, how fast will the oil be moving when it hits shore?*

These *related rates* problems obviously can be very useful. Another application of the derivative will be useful later in calculus. We will spend the rest of this chapter developing this new application. First, though, we need a definition that could have been encountered earlier.

Definition 184. *A function f is said to be* **continuous at** $x = a$ *if* $\lim_{x \to a} [f(x)] = f(a)$. *If f is continuous at every point in its domain, we say it is a* **continuous function**.

It will be proven in a later course (elementary analysis?) that any function f that is differentiable at $x = a$ is also continuous at $x = a$. Therefore, any statement that assumes a function f is differentiable inherently includes the assumption that f is continuous.

Problem 185. *Complete the statement: Suppose p is a differentiable function with $a < b$ and $p(a) = p(b)$. Then there must be some value of c between* *a and b such that $p'(c)\ldots.$*

Problem 186. *Verify the previous statement for the function $p(x) = x^3 - 3x^2 - 4x$ with $a = 0$ and $b = 4$.*

Problem 187. *Are there any horozontal tangent lines on the graph of $q(x) = x^5 - 3x^4 - 3x^3 - 3x^2 - 4x$ between $x = -1$ and $x = 0$?*

Problem 188. *If $f(x) = 4x^5 + x^3 + 7x - 2$, find the number of roots of f:*

(a.) *Compare $f(0)$ with $f(1)$. What does this say about the number of roots of f?*

(b.) *Use Problem 185 to show that f has exactly one real root.*

Problem 189. *Suppose $f(x)$ is a function that is continuous on the interval $[a,b]$. Define the function h as follows:*

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

(a.) *What is $h'(x)$?*

(b.) *What is $h(a)$?*

(c.) *What is $h(b)$?*

(d.) *What does this (and Problem 185) tell you about the behavior of the function f between a and b?*

Theorem 190. *For any function f that is differentiable on the interval $[a,b]$, there must be a point c between a and b such that $f'(c) = \ldots.$*

Problem 191. *The first two toll stations on the Hardy Toll Road are 8 miles apart. Dr. Loft's EasyPass says it took 6 minutes to get from one to the other on a Sunday drive last week. A few days later, a ticket came in the mail for exceeding the 75 mph speed limit. Can he fight this ticket?*

Problem 192. *Use Problem 190 to show that if $f'(x) = 0$ for all $x \in [a,b]$, then f must be a constant function on $[a,b]$.*

Problem 193. *If f and g are differentiable functions and $f'(x) = g'(x)$ for all x, what can be said about the functions f and g?*

Problem 194. *Find all functions g that satisfy* $g'(x) = x^2 + 6x - 10$.

Problem 195. *Find all functions h that satisfy* $h'(x) = x \cdot \sin(x^2)$.

Problem 196. *Find all functions* f *that satisfy* $f'(x) = 2 \cdot f(x)$.

Problem 197. *Find all functions* f *that satisfy* $f''(x) = -f(x)$.

Chapter 7

Antiderivatives

– “*What’s the integral of* $\frac{1}{cabin}$ *with respect to cabin?*”
– “*A log cabin.*”
– “*No, a houseboat. You forgot to add the C.*”

Being able to calculate the derivative of a function f has many uses, as we have seen in earlier chapters. Alternatively, it is often valuable to find a function whose derivative is f. In this chapter we will see how this is done, and save the first application of this process for the next chapter. Many more applications will be seen next semester in Calculus II: Integral Calculus. First, a review of some of the derivatives we learned in the previous chapters.

Problem 198. *Compute the derivatives of each of the following:*

(a.) $a(x) = e^{6x}$

(b.) $b(x) = \sqrt{x^4 + x^2}$

(c.) $c(x) = 4x^2 - 8x + (3x-7)^2 + \frac{4}{x} - \frac{1}{4x^2}$

(d.) $d(x) = \tan^5(10x - 7)$

Definition 199. *Let f be a function. We say the function F is an* **antiderivative for** *f if $F'(x) = f(x)$ for all x.*

Problem 200. *Find two antiderivatives for each of the following functions:*

$$f(x) = x^2 - x + 2 \qquad g(x) = x^3 + \frac{1}{x^3} - 3 \qquad h(x) = x + \frac{1}{x}$$

Problem 201. *Find two antiderivatives for each of the following functions:*

$$f(x) = \cos x \qquad g(x) = 2x\cos(x^2) \qquad h(x) = \cos(6x)$$

Problem 202. *Find two antiderivatives for each of the following functions:*

$$f(x) = \sec^2(x) \qquad g(x) = \sec(2x)\tan(2x) \qquad h(x) = x^2\cos(x^3)$$

Problem 203. *Find two antiderivatives for each of the following functions:*

$$f(x) = e^x \qquad g(x) = e^{-12x} \qquad h(x) = 5^x$$

Problem 204. *Find ALL antiderivatives for each of the following functions:*

$$f(x) = \frac{1}{2\sqrt{x}} \qquad g(x) = \frac{1}{\sqrt{7x-6}} \qquad h(x) = \sqrt{7x-6}$$

You may have noticed that there were many correct solutions to each of these antiderivatives. In fact, if one antiderivative exists for a function, then there are actually *infinitely many* such antiderivatives. In this case, we have a special term for this collection of antiderivatives.

Definition 205. *If a function f contains an antiderivative, we call the collection of all such antiderivative functions the* **general antiderivative** *for f, and we write*

$$\int f(x)\,dx$$

Another name for this collection of functions is **the indefinite integral of** f.

Problem 206. *Integrate the following functions. That is, compute the following indefinite integrals:*

(a.) $\displaystyle\int x + \frac{1}{x}\,dx =$

(b.) $\displaystyle\int \cos(-3x)\,dx =$

(c.) $\int 10^t \, dt =$

(d.) $\int \sqrt{u} \, du =$

Before we learn a method for finding some slightly more complicated antiderivatives, let's learn a simple application.

Problem 207. *Find a function h that has derivative* $h'(x) = x^4 - x^2 + 1$ *and satisfies* $h(1) = -1$.

Problem 208. *If* $y' = 2x + \sin x$ *and* $y = 2$ *when* $x = 0$, *then write y as a function of x.*

Problem 209. *The acceleration due to the Earth's gravity is approximately 9.8 m/s*^{2}. *If an object is thrown upwards with a velocity of 10 m/s, find a function which gives the velocity of the object after t seconds. If the object is thrown from the top of a 150 m building, find a function which gives the height of the object after t seconds.*

Problem 210. *One of the lunar astronauts dropped a wrench from the top of the space module (12 meters above the surface of the moon). How long did it take for the wrench to hit the ground? [*`Wikipedia` *is an allowable resource for this problem.]*

We will now learn how to compute some slightly more complicated antiderivatives. First, recall two of the solutions to Problem 198:

$$a(x) = e^{6x} \qquad \Longrightarrow \qquad a'(x) = 6e^{6x}$$

$$b(x) = \sqrt{x^4 + x^2} \qquad \Longrightarrow \qquad b'(x) = \frac{4x^3 + 2x}{2\sqrt{x^4 + x^2}} = \frac{2x^3 + x}{\sqrt{x^4 + x^2}}$$

In each of these cases, the Chain Rule was used. As a result, the derivative of the "inside" function is a factor of the derivative. In order to reverse this process, we will need to locate (1.) an inside function u of x as well as (2.) the derivative u' as an additional *factor* of the function we are trying to integrate. (Recall that we use the verb "integrate" to mean "find the indefinite integral of.")

For example, suppose we wish to integrate the function $4x^3 \cos(x^4)$. If we were to assign the "inside" function x^4 a new name: $u(x) = x^4$. Then of course $u'(x) = 4x^3$, and we could rewrite

$$4x^3 \cos(x^4) \; = \; u'(x) \cos(u(x)).$$

Consequently, it should by now be rather easy to compute the indefinite integral by "undoing" the Chain Rule involving the function u:

$$\int 4x^3\cos(x^4)\,dx \;=\; \int u'(x)\cos(u(x))\,dx = \sin(u(x))+C = \sin(x^4)+C.$$

Let's repeat this while computing another slightly more difficult indefinite integral.

Problem 211. *Consider the function* $f(x) = 6x^5(x^6+10)^4$.

(a.) If we define $u(x) = x^6+10$*, then* $u'(x) = \cdots$

(b.) Use this to write $\int f(x)\,dx$ *in terms of the new function* u.

(c.) We may now complete the process: $\int f(x)\,dx =$

We call this particular technique the Method of Substitution: we locate a candidate for the "inside" function u, compute u', and then substitute expressions involving u and u', effectively reducing the "size" of the indefinite integral. Let's practice some more.

Problem 212. *Compute* $\int (2x-1)\sec^2(x^2-x)\,dx$*, letting* $u = x^2-x$.

Problem 213. *Compute the indefinite integral* $\int (4x^3+1)(x^4+x)^6\,dx$.

Often the substitution we need to make is not as obvious, and an additional algebraic adjustment must be performed before the substitution is made.

Problem 214. *Compute the indefinite integral* $\int x^3\sqrt{x^4+1}\,dx$.

Problem 215. *Compute the following indefinite integrals*

$$\int (\sin x)(\cos x)^3\,dx \qquad \int \frac{x}{2\sqrt{3x^2-2}}\,dx \qquad \int \tan x\,dx$$

Problem 216. *Compute the following indefinite integrals*

$$\int x\sqrt{6x^2+1}\,dx \qquad \int \frac{1}{(3x+1)^3}\,dx \qquad \int \sin(6x)\cos^3(6x)\,dx$$

Chapter 8

The Fundamental Theorem of Calculus

> *If I have seen further it is only by standing on the shoulders of giants.*
>
> – Sir Isaac Newton, 1676.

In this chapter we discover one of the most fascinating applications of the antiderivative: the Fundamental Theorem of Calculus, published in 1669 by Isaac Barrow. We currently teach calculus in the following order: limits, derivatives, integrals. But history tells us that the concepts were developed in the opposite of this order, with the concept of the definite integral first.

Definition 217. *Suppose f is a function that is positive for every x on the interval $[a,b]$. The area of that region in teh xy-plane bounded on the sides by the vertical lines $x = a$ and $x = b$, above by the graph of f, and below by the x-axis is written as*

$$\int_a^b f(x)\,dx$$

We call this the **definite integral of f from a to b**.

Problem 218. *Sketch the graph of $f(x) = 3x$. Find $\int_0^2 f(x)\,dx$, $\int_2^3 f(x)\,dx$, and $\int_0^3 f(x)\,dx$.*

Problem 219. *Sketch the graph of $h(x) = \sqrt{x}$. Estimate $\int_0^2 h(x)\,dx$ using a triangle and a trapezoid, both with bases on the x-axis.*

Problem 220. *What could be done in Problem 219 to make the estimate a better one?*

If we wish to find the sum of many terms that all have the same general form, we have some special notation. For example, all even integers take the same form $2n$ for some integer n. If we wish to add up all the reciprocals of even positive integers, then we would write:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{n^2}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

We may use this notation to write an expression for the estimate of the area under a graph as follows.

Problem 221. *Consider the graph of $f(x)=\sin x$ between $x=0$ and $x=\pi$. We will divide this region into several (let's say n) narrow vertical rectangles – all with the same width Δx. Now choose any number in each subinterval. Label these numbers c_i. For example, the number we would choose in the fifth subinterval would be called c_5.*

(a.) We may use each c_i to define the height of a rectangle. What will the dimensions of these rectangles be?

(b.) Write down an expression for the sum of the areas of these rectangles.

(c.) How could this sum be a better approximation for the actual area $\int_0^{\pi}\sin x\,dx$?

Problem 222. *Write down an expression for a good approximation for the area under the curve $g(x)=x^3$, between $x=1$ and $x=5$.*

Problem 223. *Write down an expression (involving a limit) for the actual area under the curve $h(x)=\tan x$, between $x=0$ and $x=\frac{\pi}{4}$.*

Definition 224. *For a function $f\geq 0$, the definite integral $\int_a^b f(x)\,dx$ can be found by the limit ....*

Problem 225. *What would be different about this process if $f(x)<0$ for all x between a and b? That is, what could we say about the number $\int_a^b f(x)\,dx$ if the function values for f were negative on the interval $[a,b]$?*

Problem 226. *Suppose it is known that $\int_0^{\pi}\sin x\,dx=2$. What can be said about the value of $\int_{-\pi}^{0}\sin x\,dx$?*

Problem 227. *Compute the following using the graphs of each function:*

$$\int_{-4}^{4} x^3\,dx \qquad \int_{-\pi/4}^{\pi/4} \tan x\,dx \qquad \int_{-3}^{0} 2x\,dx$$

Problem 228. *Let f be a function with $x = a$ some number in the domain of f. Define a new function A as follows: the number $A(t)$ is the signed area bounded by the x-axis, the vertical lines $x = a$ and $x = t$, and the graph of f. That is, $A(t) = \displaystyle\int_a^t f(x)\,dx$. Let's find the derivative of this new function A.*

(a.) Describe $A(t+h)$. Write down an expression for $A(t+h)$.

(b.) Can the difference $A(t+h) - A(t)$ be simplified? How?

(c.) Try to write the difference quotient $\dfrac{A(t+h) - A(t)}{h}$ without using the integral.

(d.) What does this say about the relationship between the two functions A and f?

Theorem 229 (The Fundamental Theorem of Calculus, Part A). *For any continuous function f and any number a in its domain, …*

Problem 230. *Let $f(x) = \cos x$. Define $A(t) = \displaystyle\int_0^t \cos x\,dx$. Then*

$$A'(t) = \underline{\qquad\qquad} \quad \textit{and} \quad A'(4) = \underline{\qquad\qquad}$$

Problem 231. *Let $G(t) = \displaystyle\int_1^t 5x^3 + 7\,dx$. Use Theorem 229 to compute G'.*

Problem 232. *Let $H(t) = \displaystyle\int_1^{\sqrt{t}} 6\sin x\,dx$. Use Theorem 229 to compute H'.*

Problem 233. *Suppose f is a continuous function with $x = a$ in its domain. Part A of the Fundamental Theorem of Calculus gives us one antiderivative A for f. If we were able to find* another *antiderivative F for f, then write an equation describing the relationship between F and A.*

Evaluating the equation from Problem 233 at both $x = a$ and $x = b$ tells us something remarkable about the relationship between the definite integral and the indefinite integral of a function f.

Theorem 234 (The Fundamental Theorem of Calculus, Part B). *If F is <u>any</u> antiderivative for f, then $\int_a^b f(x)\,dx = \ldots$.*

Problem 235. *Compute the following definite integrals using Part B of the Fundamental Theorem of Calculus.*

$$\int_0^{\pi} \sin x\,dx \qquad \int_0^2 x^2\,dx \qquad \int_0^4 \sqrt{x}\,dx$$

Problem 236. *Compute the following definite integrals.*

$$\int_{-\pi}^{\pi} 2x\sin(x^2)\,dx \qquad \int_{-2}^{2} x^2\,dx \qquad \int_0^1 2x\sqrt{x^2+1}\,dx$$