A GENERALIZATION OF SURGERY TO CELLULAR COMPLEXES

A Thesis

Presented to

The Faculty of the Department of Mathematics

Sam Houston State University

In Partial Fulfillment

of the Requirements for the Degree of

Masters of Science

---

by

Ryan J. Gueli

May, 2018

A GENERALIZATION OF SURGERY TO CELLULAR COMPLEXES

by

Ryan J. Gueli

APPROVED:

Brian M. Loft, PhD.
Committee Director

Brandy Doleshal, PhD.
Committee Member

Taylor Martin, PhD.
Committee Member

Damon Hay, PhD.
Committee Member

John B. Pascarella, PhD.
Dean, College of Sciences
& Engineering Technology

DEDICATION

This thesis is dedicated to my former high school calculus teacher Bruce Reopolis. I would never have become a math major if it weren't for you. You taught me that being a nerd is cool, and I really "nerded out" with this thesis. "Here's looking at you kid."

ABSTRACT

Gueli, Ryan J., A Generalization of Surgery to Cellular Complexes. Masters of Science in Mathematics, March, 2018, Sam Houston State University, Huntsville, Texas.

In the 1990s Robin Forman carefully generalized Morse theory to cellular complexes. In this thesis we are concerned with the connection Morse theory provides in surgery theory. There exists a theorem linking surgery on embedded spheres to cobordisms whose proof relies on smooth Morse theory. Therefore, we examine if a similar theorem exists on cellular complexes whose proof relies on discrete Morse theory. In order to accomplish this we will provide a way to perform surgery on cellular complexes as well as a possibility for defining cellular cobordism.

KEY WORDS: Differential Geometry, Smooth Morse Theory, Discrete Morse Theory, Surgery, Cobordism, Cellular Surgery, Locally Cellular Cobordant.

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my research advisor Dr. Brian Loft. Dr. Loft took me under his wing in my first semester of graduate school and has been there for me ever since. Whether it is a research question, a question from another class, or a question about life, Dr. Loft has always been there for me. Thank you for everything you have done for me.

Next, I must thank my graduate advisors. First, I must thank Dr. Ken Smith, thank you for taking a chance on me as a student. I feel like I have taken the opportunity you gave me and ran with it. I wouldn't be where I am today if it weren't for the chance you gave me. Also, I must thank Dr. Edward Swim. I don't think I can count on one hand how many times I would come into your office asking for help in some way shape or form. Whether it was what class to take, when deadlines are do, or where to get good pizza in Texas, you always were willing to help with open arms. Thank you for all you have done for me.

I would also like to thank all my teachers who pushed me to my best throughout my Masters career. My analysis professor Daniel, you have pushed me to learn a subject I never thought I would even comprehend. Also thank you for making me LaTex so much homework. I am positive this thesis wouldn't have made it this far if it weren't for all the practice. To Dr. Luis Garcia thank you for listening to my countless questions in algebra as well as helping me start graduate chalk talks. I feel as if you always believed in me and that meant the world to me. To Dr. Brandy Doleshal, you helped me see topology from a completely different viewpoint than I am used to seeing. Thank you for pushing me to try learning in new ways, as well as seeing problems in a different light. Also, thank you for being on my thesis committee, I know life can be hectic so taking out the time to help me with this process means so much to me. Dr. Malandro, any combinatorics faced in my research was a breeze thanks to your class. Also, I must have you know that to this day when I teach I try to emulate your style because I definitely do not write enough. To Dr. Hay, thank you for being on my thesis committee and more importantly for helping me become a leader in our student chapter of AMS. Also, thank you for always being a caring individual giving me somewhere to vent my frustrations. I most likely would have had a melt down if it weren't for you. Lastly, thank you Dr. Martin. First and foremost for being on my thesis committee but also for helping me get through my first semester of teaching. I definitely wasn't ready for some of the dilemmas I faced in the classroom, thank you for helping me through.

I would like to thank the rest of the math department at Sam Houston State University. Thank you to the lectures Cathy Lockwood, Casandra Wright, and Cheri Huggins, the office workers Jordan Story and Ms. Lisa, as well as the undergraduates at the desk. Thank you to the chair Dr. Melinda Holt for helping support AMS and all our events.

Thank you to my fellow graduate students Josh, Neil, Syrian, Valentine, Kasun, Mai, Sulimon, Sulani, Joey, Katie, Salem, Chamika, Kevin, Kerrie, Vy, Brittany, Mckenzie, Nuwanthi, and all the stats graduate students. You have all become my family. We all did this together. Thank you for making me feel at home. Thank you to my partner in crime Rutger Yager. Who would of thought we would've become so close after playing Call of Duty, and eating pizza at an AMS social. We have worked together through some of the hardest problems, through some of the most difficult class and experiences, but we did it together. I honestly couldn't of done any of this without you by my side. These past two years we became family, I love you like a brother thank you for everything.

I would like to thank all my friends I have made over the years. Those back in Buffalo, and those here in Texas, all your support is what keeps my motor going everyday.

Thank you to my wonderful parents. I know it was hard letting me go all the way to Texas but look at how well it turned out. Thank you for your continued support, I wouldn't be here without you both. Thank you to my sister and her husband, thank you for always making me feel smarter by never being shocked when I did well in school. I always noticed and always appreciated every moment. Thank you to my brother and his wife and two amazing kids, you all mean the world to me. When I first moved to Texas I was alone, but you all took me under your roof and loved me unconditionally, especially letting me do loads of laundry. Thank you to my entire family, I love you all so much and thank you for always believing in me.

Finally, to my wonderful girlfriend Mandi. We have had tons of ups and downs but I wouldn't be here if it wasn't for you pushing me harder than anyone else. You always tell me "You're Ryan Gueli, You got this!!" and that is exactly why I am writing this thesis today. You always supported me, always fought for me and constantly helped me, even when it was a burden to you. This next journey is going to be hard on me but I am not worried knowing I have you by my side. I love you and thank you for everything.

-Ryan Gueli

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

Introduction

This thesis examines a connection between the discrete and the continuous in mathematics. In most instances, there are natural connections between both discrete and continuous mathematics. Sometimes when working in a discrete setting, there are natural continuous settings that also apply. For instance, consider the discrete probability model and notice as the size of the population increases it approaches the continuous probability distribution model. Other discrete analogues of continuous mathematics are discrete calculus, discrete Fourier transforms, discrete geometry, discrete dynamical systems and most importantly for our purposes, discrete Morse theory.

The discrete and continuous connection we value in this thesis is between discrete and smooth Morse theory. Smooth Morse theory has several applications, but for the purposes of this thesis, we focus on its application to surgery theory. A classic theorem relates surgery theory to cobordism, in which the proof of this theorem relies on smooth (or continuous) Morse theory. Therefore, since we know there is a discrete analogue for smooth Morse theory we may ask, "Is there a notion of discrete surgery, and if so is there a notion of discrete cobordisms?"

In this thesis we show how through discrete Morse theory that we can extend surgery to cellular complexes. This extension provides a way to construct a discrete cobordism. Through construction, we found our generalizations will preserve the same results found in the smooth case.

We will begin by introducing some basic topology definitions, and defining smooth Morse theory as well as give some simple results in the field. We will do the same with discrete Morse theory before moving into surgery on embedded spheres. We will explore examples of surgery in the continuous case, and then will define cobordism as well as list examples. After defining cobordism, we will state a classic theorem that shows the connection between surgery theory and cobordism, whose proof will rely on smooth Morse theory. We will finish by sharing our algorithm for generalizing surgery to CW-complexes, as well as defining discrete cobordism.

CHAPTER 2

Background Topology

In order to define smooth and discrete Morse theory, we must first familiarize ourselves with some basic definitions in topology. We begin by defining basic point set topological ideas, then move into classifying "sameness" using homeomorphisms and homotopy equivalences. Then we will define a topological manifold as well as a smooth manifold, before finally defining how to create a CW-complex.

2.1 Point-set Topology Definitions

First, we introduce a topology.

Definition 1 (Topology). A topology on a set X denoted $\mathscr{T}$, is a nonempty collection of subsets of X, called open sets denoted O_n where $n \in \mathbb{N}$, such that the following are true :

1. If $\{O_1, O_2, \ldots, O_n, \ldots\} \in \mathscr{T}$ then $\bigcup_i O_i \in \mathscr{T}$ where $O_i \in \{O_1, O_2, \ldots, O_n, \ldots\}$.
2. If $\{O_1, O_2, \ldots, O_n, \ldots\} \in \mathscr{T}$ then $\bigcap_{i=1}^{n} O_i \in \mathscr{T}$ where $O_i \in \{O_1, O_2, \ldots, O_n, \ldots\}$.
3. $X \in \mathscr{T}$ and $\emptyset \in \mathscr{T}$

A set together with a topology is called a topological space.

For purposes of this thesis the topology we consider is the *standard topology* on a space. The standard topology defines the subsets of $\mathbb{R}^n$ for $n \in \mathbb{N}$ as the open sets in $\mathscr{T}$.

Example 2 (Standard Topology on $\mathbb{R}$). Let $X = \mathbb{R}$ and let $\mathscr{T} = \{\emptyset, B, \mathbb{R}\}$ where B is the collection of all open intervals of $\mathbb{R}$. This is called the standard topology on $\mathbb{R}$.

Now that we know what it means for sets in our space to be defined open, we can define a closed set.

Definition 3 (Closed). A set X is closed if its complement $X^c \in \mathscr{T}$.

Now we have the necessary and sufficient definitions in order to define what a continuous function is on a topological sense.

Definition 4 (Continuous Function). Let X and Y be topological spaces, and define a function $f : X \to Y$. Then we say that f is a continuous function if for each open subset $V \subseteq Y$, $f^{-1}(V)$ is open in X.

This definition is consistent with the definition of continuous in an analytical sense. Consider the following example:

Example 5. Let $X, Y = \mathbb{R}$ with the standard topology, and consider the constant function $f : X \to Y$ where $f(x) = y$ for all $x \in X$. We know from analysis that the constant function is continuous, and we will show this is also true for our topological definition of continuous. Let $V \subseteq Y$ be open, then $f^{-1}(V) = \emptyset$ if $y \notin V$ and $f^{-1}(V) = X$ if $y \in V$. We know since X is a topological space, both $\emptyset$ and X are open. Therefore, f is continuous.

Continuous functions are necessary in defining a homeomorphism as well as homotopy equivalences. We also must recall what it means for a space to be compact. Later in this thesis, we will refer to compact manifolds.

Definition 6 (Open Cover). Let X be a topological space and let $\mathscr{A} = \{O_1, O_2, \ldots, O_i, \ldots\}$ be a collection of open sets with $O_i \subseteq X$ for each i. Then $\mathscr{A}$ is an open cover of X if $\cup_i O_i = X$.

Definition 7 (Compact). A topological space X is compact if every open covering of X has a finite subcovering.

That is if $\mathscr{A}$ is an open cover of X, then if there exists $\{O_1, O_2, \ldots, O_n\} \subseteq \mathscr{A}$ where $\cup_i^n O_i = X$, then X is compact.

This definition can be quite difficult to show sometimes, therefore we state the following theorem without proof, to help more easily define when a space is compact.

Theorem 8. *A space $X \subseteq R^n$ is compact if and only if X is closed and bounded. [1]*

When we refer to a space being compact in this thesis, our space will be a subset of Euclidean space. Therefore, Theorem 8 more easily allows us to analyze when a space is compact.

Example 9. Consider a circle or S^1 defined by $\{(x,y) \mid x^2 + y^2 = 1\}$. We will prove this space is compact using Theorem 8. First, we note that by our definition, S^1 is a subset of $\mathbb{R}^2$. When looking at the complement of $\{(x,y) \mid x^2 + y^2 = 1\}$ we have $\mathbb{R}^2$ without S^1 around the origin. We know this space is open, and therefore S^1 is closed. Also we note that S^1 is bounded by definition. Therefore, S^1 is closed and bounded; thus by Theorem 8, S^1 is compact.

Compactness is an invariant of topological spaces. Another invariant we need is that of connectedness. We will use connectedness as a homotopic invariant later in this thesis.

Definition 10 (Connected). A space X is <u>connected</u> if whenever $X = A \cup B$ of two nonempty subsets A, B, then $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$. We note that $\overline{A}, \overline{B}$ is the <u>closure</u> of A and B respectively.

We now state a theorem without proof that allows for us to more easily show when X is connected.

Theorem 11. *Let X be a topological space then X is connected if and only if X cannot be expressed as the union of two disjoint nonempty open sets. [1]*

We say that if X can be expressed as the union of two disjoint nonempty open sets, then we can create a *separation* of X.

Example 12. Let $X = S^1$. We cannot create a separation of S^1. This is because when taking open sets on S^1, we are taking open arcs on S^1. Therefore, if we tried to make a separation we would either not obtain the entire space, or the intersection of our open sets would not be empty. Therefore, by Theorem 11, S^1 is connected.

Example 13. Let $X = S^1 \sqcup S^1$ that is, X is the disjoint union of two copies of S^1. We can create a separation of this space. Let A be one copy of S^1 and let B be the other copy. We note that A and B must be open. We see because X is the disjoint union of the circles that $A \cap B = \emptyset$. Therefore, we have provided a separation for X. Thus by Theorem 11, X is disconnected.

Now that we have recalled various point set topology definitions, we can define "sameness" of topological spaces.

2.2 Homeomorphisms

We begin by defining topological equivalence, a homeomorphism. Throughout this thesis we will refer to spaces being homeomorphic to one another, when we do this we mean the following :

Definition 14 (Homeomorphism). Let X and Y be topological spaces. A function $h : X \to Y$ is a <u>homeomorphism</u> if h is a bijective, continuous function such that $h^{-1} : Y \to X$ is also continuous. We say two sets are <u>homeomorphic</u> if there exists a homeomorphism between the sets.

The intuition of a homeomorphism is that we can stretch and morph one object into another without tears or breaks. We cannot collapse an object to a point, or cut and break an object and preserve a homeomorphism. We will consider the following examples to better understand the notion of a homeomorphism.

Example 15. Let X be a square where $X = \{(x,y) : |x| = 1, |y| \leq 1\} \cup \{(x,y) : |x| \leq 1, |y| = 1\}$. Let Y be the circle where $Y = \{(x,y) : x^2 + y^2 = 1\}$. Consider Figure 1 of Y inscribed within X. We notice that in each quadrant we can normalize points

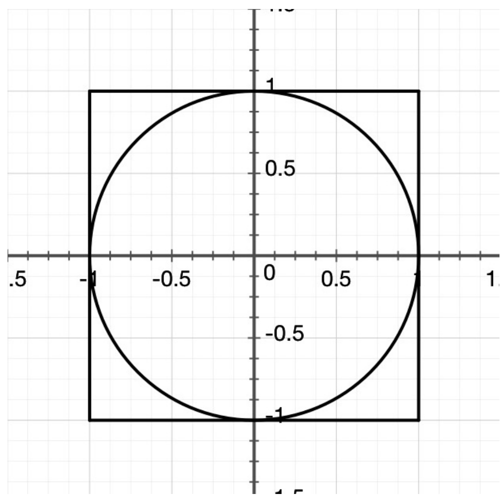

Figure 1: A circle inscribed in a square

on X to points on Y using the function $f : X \to Y$ defined by $(x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$. Through basic analysis, we know the function that normalizes points is a continuous bijective function whose inverse is also continuous. Therefore, X and Y are homeomorphic. That is a square and a circle are homeomorphic.

Example 16. We can further Example 15 by letting X be the unit cube and Y the unit sphere. As before we have the sphere embedded in the cube, and again define a function that normalizes points of the cube onto the sphere. Just like Example 15 the function will be a bijective continuous function with a continuous inverse, thus a cube and sphere are homeomorphic to one another.

So far we have presented numerous spaces that are homeomorphic to one another. However there exists spaces that are not homeomorphic to one another thus illustrating limits to homeomorphisms.

Example 17. Let $X = \{0\}$ and $Y = [0,1]$. Then any $f : X \to Y$ is not a homeomorphism since any f would not be injective.

Example 18. Let $X = [0,1]$ and $Y = S^1$. Consider the function $f : X \to Y$ given by $f(x) = e^{2\pi ix}$. It is clear that this function is not injective since $\{0\}$ and $\{1\}$ are mapped to the same point on S^1. Therefore, we see that f is not a homeomorphism between X and Y.

Example 19. Let $X = S^2$ and $Y = T^2$ (the torus). X and Y are not homeomorphic to one another. However, we do not have the necessary definitions to prove this yet. We will prove this later in the chapter.

2.3 Homotopy Equivalence

We now define a weaker notion of topological equivalence called a homotopy equivalence.

Definition 20 (Homotopy). Let f and f' be continuous maps from X to Y. We say f is homotopic to f', denoted $f \simeq f'$, if there is a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = f'(x)$. The map F is called a homotopy.

This definition says that two maps f and f' are homotopic, if we can find a map F that takes a point $x \in X$ and identifies that point through a deformation from $f(x)$ to $f'(x)$. We can imagine for $x \in X$ that as we let t increase from 0 to 1, we are traveling from $f(x)$ to

$f'(x)$, therefore it is convenient to imagine the interval as a time-lapse between functions. Consider the following example:

Example 21. Let $f:[0,1]\to[0,1]$ by $f(x)=x$, for all $x\in[0,1]$. Let $f':[0,1]\to[0,1]$ by $f'(x)=0$, for all $x\in[0,1]$. We then define a homotopy $F:[0,1]\times I\to[0,1]$ by $F(x,t)=(1-t)x$. We see that $F(x,0)=(1-0)x=x=f(x)$, also we see $F(x,1)=(1-1)x=0x=0=f'(x)$. Finally, we note that $F(x,t)$ is a polynomial function, and by basic analysis we know the function is continuous. Therefore, we say f and f' are homotopic to one another where $F(x,t)$ is the homotopy.

Now using this notion of homotopy we have the following definition :

Definition 22 (Homotopy Equivalence). Two spaces X and Y have the same homotopy type, or are homotopy equivalent, if there exists maps $f:X\to Y$ and $g:Y\to X$ such that $g\circ f\simeq 1_X$ and $f\circ g\simeq 1_Y$.

That is, two spaces are homotopy equivalent if when we compose two maps $f:X\to Y$ and $g:Y\to X$ we obtain the identity homotopy. A more intuitive way to analyze this definition is that two spaces are homotopy equivalent if one can be continuously deformed into the other. At first glance, one might ask what is the difference between a homotopy equivalence and a homeomorphism? With a homotopy equivalence, we again have that shrinking and stretching (without tearing) allowed, however parts of the space may be contracted to a point. Recall that in the definition of a homeomorphism, contracting down to a point was prohibited. Consider the following examples:

Example 23. First note that any homeomorphism is a homotopy equivalence. That is because if $f : X \to Y$ is a homeomorphism then letting $g = f^{-1}$ we trivially have a homotopy equivalence.

Example 24. Let $X = [0,1]$ and $Y = \{0\}$. Also let $f : [0,1] \to \{0\}$ by $f(x) = x$, and $g : \{0\} \to [0,1]$ by $g(0) = 0$.

We see that $(f \circ g) : \{0\} \to \{0\}$ is the identity map, which is clearly homotopic to itself. Also we have $(g \circ f)(x) = 0$, for all x which we know by Example 21 that $(g \circ f)(x) = 0$, for all x is homotopic to the identity.

Therefore by definition we have that X and Y are in the same homotopy class. This means that a point and an interval are homotopy equivalent. Which we know in the case of homeomorphisms was not true.

We notice that the definition of homotopy equivalence expands the notion of sameness originally created by a homeomorphism. However just as in the case of homeomorphism, there are limits to homotopy equivalences.

Example 25. A circle and an interval are not homotopy equivalent. In order to obtain an interval from a circle we must cut or break the circle, which would not be a continuous mapping. Therefore just as in the case of homeomorphisms a circle and an interval are not consider the same.

Example 26. Let $X = S^1$ and $Y = S^1 \sqcup S^1$. We recall from Examples 12 and 13 that X is connected while Y is not connected. Connectedness is a homotopy invariant. Therefore X and Y are not homotopic.

Therefore, we notice that there are numerous spaces that are still not homotopy equivalent to one another.

Now that we have background for classifying sameness, we will now define a manifold. Manifolds will be heavily referred to in smooth Morse theory.

2.4 Manifolds

Definition 27 (Neighborhood). A neighborhood V of a point p is a set $V \subseteq X$, such that there exists $U \in \mathscr{T}$ where $p \in U \subseteq V$.

Definition 28 (Manifold). A n -dimensional manifold, M^n is a space so that for each point $x \in M^n$ there exists a neighborhood U containing x that is homeomorphic to $\mathbb{R}^n$. That is, a manifold is a space that is locally Euclidean.

Example 29. We will first show that $\mathbb{R}^n$ is a manifold. Let $p \in \mathbb{R}^n$ with neighborhood U such that $p \in U$. We know that open sets of $\mathbb{R}^n$ are open n-disks by definition. Therefore $U \cong D^n$. We then can define $f : \mathbb{R}^n \to U$ by $f(x) = \frac{x}{\|x\|+1}$. It is a straightforward exercise to show that f is a homeomorphism. Therefore we see that a neighborhood around a point in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$. Therefore $\mathbb{R}^n$ is a manifold.

Example 30. Another example of a manifold is the (n)-sphere that is defined to be the boundary of the $(n+1)$-disk. That is,

$$S^n = \left\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \ : \ x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\right\}$$

In order to show this, we consider the stereographic projection from R^{n+1} to S^n. That is $S^n = U_N \cup U_S$ where $U_N = S^n / \{(0,0,\ldots,0,1\}$ and $U_S = S^n / \{(0,0,\ldots,0,-1)\}$ which are both open under the induced topology from the standard topology on $\mathbb{R}^{n+1}$.

We then define $\phi_N : U_N \to \mathbb{R}^n$ by $\phi_N(x_1, x_2, \ldots, x_{n+1}) = \left(\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right)$

and $\phi_S : U_S \to \mathbb{R}^n$ by $\phi_S(x_1, x_2, \ldots, x_{n+1}) = \left(\frac{x_1}{1+x_{n+1}}, \frac{x_2}{1+x_{n+1}}, \ldots, \frac{x_n}{1+x_{n+1}}\right)$. Again it is straightforward exercise to show ϕ_N and ϕ_S are homeomorphisms. Therefore, we have a neighborhood around any point in S^n is homeomorphic to $\mathbb{R}^n$. Therefore, S^n is a manifold.

Example 31. Through Examples 29 and 30 we have examples of one dimensional manifolds are S^1 or $\mathbb{R}$. In these cases we let $n = 1$ in Examples 29 and 30.

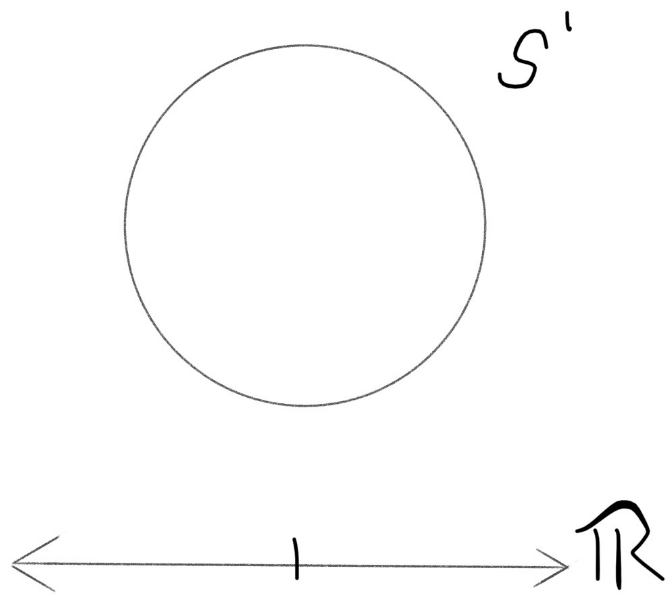

Figure 2: Manifolds of dimension 1

Example 32. Manifolds of dimension two are usually referred to as surfaces. First, we know that by Examples 29 and 30 that S^2 and $\mathbb{R}^2$ are manifolds. We also know that T^2 is a manifold. We will provide intuition why this is true. Let $p \in T^2$, and let U be a neighborhood where $U \subset T^2$ with $p \in U$. Then if we were to "cut out" U from T^2 we would notice that it flattens. We note that U is an open disk in $\mathbb{R}^2$ and thus homeomorphic to $\mathbb{R}^2$. This gives us that T^2 is locally Euclidean and thus a manifold.

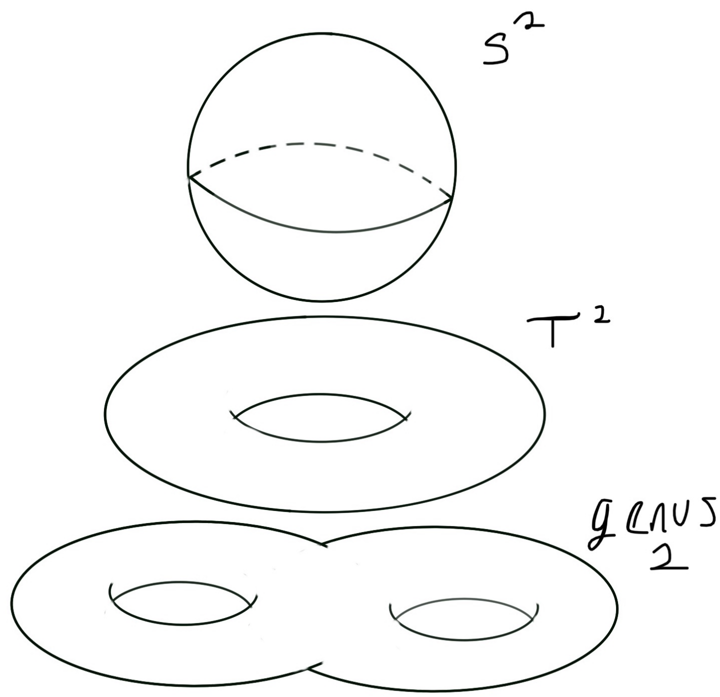

Figure 3: Manifolds of dimension 2, or surfaces

In smooth Morse theory we will be working on smooth manifolds. However, we must first define what a coordinate chart is, and then we can move onto defining a smooth manifold.

Definition 33 (**Coordinate Charts).** Let M be a topological space. A homeomorphism $\phi : U \to V$ of an open set $U \subset M$ onto an open set $V \subset \mathbb{R}^d$ will be called a coordinate chart.

Definition 34 (**Smooth Manifold).** [10] Let M be a topological space. Let A be an indexing set, where $\alpha, \beta \in A$, with $\alpha \neq \beta$. We say that M is a smooth manifold if we can equip M with a differentiable structure C^∞. A differentiable structure, or smooth structure on M is a collection of coordinate charts $\phi_\alpha : U_\alpha \to V_\alpha$, where $V_\alpha \subseteq \mathbb{R}^d$ such that :

1. $M = \cup_\alpha U_\alpha$

2. For every α, β the change of local coordinates $\phi_\beta \circ \phi_\alpha^{-1}$ is homeomorphism.

3. If for every α, β the change of local coordinates $\phi_\beta \circ \phi_\alpha^{-1}$ is of class C^∞ then M is of class C^∞

We see that manifolds are restrictive. Therefore we define how to create cellular complexes which allows us to generalize spaces. With cellular complexes we may construct both manifolds and spaces that are not manifolds.

2.5 CW-Complexes

A CW-complex is a collection of *n-cells*.

Definition 35 ($n-cell$ [5])**.** An $\underline{n-cell}$ (denoted e^n) is homeomorphic to the interior of an n-disk, where the n-disk, D^n is defined to be the collection of points in $\mathbb{R}^n$ which satisfy the inequality

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1.$$

Some simple examples of cells are the 0-cell which a point, a 1-cell which is homeomorphic to a interval and a 2-cell which is homeomorphic to a disk. Also, it is important to note that throughout this paper when we refer to the word *face*, that we are referring to the boundary of a cell. For example the face of a 2-cell is a collection of 1-cells. We denote a face by $\tau \succ \sigma$ where τ is a face of σ.

Now using this definition of n-cells, we create cellular complexes. We define an n-dimensional complex as follows :

Definition 36 (***n*-dimensional Cellular Complex (or *n*-complex)[5]).** Let X be a collection of k-cells, for $k = 0, 1, \ldots, n$.

1. The collection of 0-cells is called <u>the 0-skeleton of the complex</u>.

2. Attach the set of 1-cells, one at a time, to the 0-skeleton as follows: identify (or glue) each edge of each 1-cell to one of the 0-cells. The resulting object is called the 1-skeleton of the complex.

3. Repeat for $k = 2, 3, \ldots, n$: For each k-cell, identify this boundary to either (a.) a homeomorphic copy of S^{n-1} in the (n - 1)-skeleton or (b.) a 0-cell in the 1-skeleton.

It is important to note that when we attach a p-cell to a complex X, we denote this as $e^p \cup X$. However, when we use this notation we do not mean the classic definition of union. We have that there is an applied attaching map. That is we really mean $e^p \cup_f X$, where $f : S^p \to X$.

With the definitions we have provided we can construct the manifolds we listed in the previous section.

Example 37. The 2-sphere, S^2 can be created using one e^0, one e^1 and two e^2. First, attach e^0 and e^1 to create a circle. Then, we attach the two e^2 to the circle to create a sphere. Therefore, we say S^2 has the following cellular decomposition $e^0 \cup e^1 \cup e^2 \cup e^2$.

Example 38. The torus can be created using one e^0, two e^1 and one e^2. We can attach the two e^1 to e^0 to obtain two circles who share exactly one point. Then, attach the e^2 to the resulting 1-skeleton to obtain the torus. Therefore, we say the torus has the following cellular decomposition $e^0 \cup e^1 \cup e^1 \cup e^2$.

The cellular decomposition of a complex provides useful information that helps distinguish spaces. In order to do this we have the following definition.

Definition 39 (Euler Characteristic). Let Y be an n-dimensional cellular complex. We denote the number of j-cells in the complex as c_j. Then the Euler Characteristic is defined as :

$$\chi(Y) = \sum_{j=0}^{n} (-1)^j c_j$$

Example 40. For the 2-sphere, we saw in Example 37 that our decomposition was $e^0 \cup e^1 \cup e^2 \cup e^2$. Therefore,

$$\chi(S^2) = c_0 - c_1 + c_2 = 1 - 1 + 2 = 2$$

Therefore, the Euler characteristic of S^2 is 2.

Example 41. For a torus or T^2, we saw in Example 38 that our decomposition was $e^0 \cup e^1 \cup e^1 \cup e^2$. Therefore,

$$\chi(T^2) = c_0 - c_1 + c_2 = 1 - 2 + 1 = 0$$

Therefore, the Euler characteristic of the torus is 0.

We will now state but not prove a theorem that will show the usefulness of the Euler characteristic.

Theorem 42. *If X and Y are closed surfaces with $\chi(X) \neq \chi(Y)$ then X is not homeomorphic to Y. [7]*

Example 43. Recall in Example 20 we wanted to show that S^2 and T^2 were not homeomorphic to one another. We will prove this using Theorem 42.

Let $X = S^2$ and let $Y = T^2$. We know by Examples 40 and 41 that $\chi(X) = 2$ and $\chi(Y) = 0$. We have that $\chi(X) \neq \chi(Y)$, and therefore by Theorem 42, $S^2 \ncong T^2$.

Therefore, we see that the Euler characteristic can serve to help classify sameness of spaces.

Cellular complexes allow a great degree of freedom. We can, create spaces that are not manifolds. Using the definitions above we can construct two cubes who share exactly one 0-cell, which is a space that is not a manifold. We would like to be able to classify these spaces.

Now that we have topological knowledge, we will move forward into smooth Morse theory. In the next chapter, we will give an overview of smooth Morse theory since it will later serve as a tool to prove a theorem involving surgery theory.

CHAPTER 3

Smooth Morse Theory

Smooth Morse theory heavily relies on the critical point of a smooth function on a manifold. We will classify the critical points of a smooth function in a particular way in order to define a Morse function on a space. Therefore, we begin this chapter with defining what a critical point is and how to classify critical points. Then, we will define the Hessian matrix and show how it is used in defining a smooth Morse function on a manifold. Finally, we will list some classical results in Morse theory that directly correlate to our focus on surgery theory.

3.1 Critical Points and their Index

In this chapter, when we refer to a manifold, we will be referring to smooth manifolds as defined in the previous chapter. We first define a smooth map.

Definition 44 (Smooth Map). [10] Let M be a smooth manifold. A function $f : M \to \mathbb{R}$ is smooth if for each point $x \in M$ f is of class C^∞ with respect to a local coordinate system; that is if $\phi_\alpha^{-1} : V_\alpha \to U_\alpha$ is a smooth coordinate system at x, then the composition

$f \circ \phi_\alpha^{-1} : V_\alpha \to \mathbb{R}$ is of class C^∞.

Now that we know what we mean when we say smooth map, we will define a critical point, and explain how to find them for a given function.

Definition 45 (Critical Point, Critical Value). Consider the function $f : \mathbb{R}^n \to \mathbb{R}$, then the point $\vec{p} = (x_1, x_2, \cdots, x_n)$ is a critical point of f if

$$\frac{\partial f}{\partial x_i}(p) = 0 \quad \text{for all } \ i = 1, 2, \ldots, n.$$

If p is a critical point for f, we call the value $f(p)$ the critical value of p.

Example 46. Consider the function $f(x,y) = x^2 + \cos(2y)$ on $M = \mathbb{R}^2$.

We see

$$\frac{\partial f}{\partial x} = 2x \text{ and } \frac{\partial f}{\partial y} = -2\sin 2y$$

Notice that $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ only at the origin of $\mathbb{R}^2$. That means that the origin $(0,0)$ is a critical point of f on M.

Also, we notice that $f(0,0) = 1$. Therefore, 1 is the critical value for the critical point $(0,0)$.

Example 47. Consider the function $f(x,y) = x^4 - 5x^2y^3$ on $M = \mathbb{R}^2$.

We see

$$\frac{\partial f}{\partial x} = 4x^3 - 10xy^3 \text{ and } \frac{\partial f}{\partial y} = -15x^2y^2$$

Notice that $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ only at the origin of $\mathbb{R}^2$. That means that the origin $(0,0)$ is a critical point of f on M.

Also, we notice that $f(0,0) = 0$. Therefore, 0 is the critical value for the critical point $(0,0)$.

We now define a way to classify critical points. This classification is essential to defining a Morse function. In order to classify critical points we first need to define the Hessian of a function.

Definition 48 (Hessian). [8] Let $f : M \to \mathbb{R}$ be a smooth function on the n-dimensional manifold M and let p be a point on M. Choose local coordinates $(x_1, x_2, \ldots, x_n)$ near p. The Hessian of f at p is the $n \times n$ matrix of second partial derivatives of f at p. We denote

$H_f(p)$ its determinant

$$H_f(p) = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$$

Example 49. Consider the function $f(x,y) = x^2 + \cos(2y)$ on $M = \mathbb{R}^2$.

We know from before that $(0,0)$ is a critical point of f on M.

Computing the Hessian we have that

$$H_f(x,y) = \det\begin{bmatrix} 2 & 0 \\ 0 & -4\cos(2y) \end{bmatrix} = -8\cos(2y)$$

We know then at our critical point $(0,0)$ that $H_f(0,0) = -8$.

Example 50. Consider the function $f(x,y) = x^4 - 5x^2y^3$ on $M = \mathbb{R}^2$.

We see know from before that $(0,0)$ is a critical point of f on M.

Computing the Hessian we have that

$$H_f(x,y) = \det\begin{bmatrix} 12x^2 - 10y^3 & -30xy^2 \\ -30xy^2 & -30x^2y \end{bmatrix} = -900x^2y^4 - 360x^4y + 300x^2y^4$$

We know then at our critical point $(0,0)$ that $H_f(0,0) = 0$.

We are now able to compute the Hessian of a function, therefore it is important to show why we care about the value of the Hessian at a critical point. This value provides a classification of critical points as *degenerate* or *nondegenerate* critical points.

Definition 51 (Degenerate & Nondegenerate Critical Points). [9] A critical point p of $f : M \to \mathbb{R}$ is <u>nondegenerate</u> if $H_f(p) \neq 0$. Since this condition is independent of the coordinate system of the coordinate system near p, this notion is well-defined. Moreover, since coordinate changes are described by multiplication by a matrix and its transpose, the sign $H_f(p)$ is independent of the coordinate system.

With degenerate critical points we have infinitely many directions of increase and decrease. For our purposes, we want to avoid degenerate critical points in order to define a Morse function on a space. However, we will discuss this in more detail later.

The Hessian provides more information than if a critical point is degenerate vs nondegenerate. It also provides the index of a critical point which plays a role in the handlebody decomposition of a space that we will touch on later. In order to find the index of a critical point we will need to recall the definition of eigenvalues and eigenvectors of a matrix.

Definition 52 (Eigenvalues and Eigenvectors). Let A be an $n \times n$ matrix with entries in $\mathbb{R}$. The nonzero vector $\vec{v}$ is an eigenvector of A if there is a number λ such that

$$A\vec{v} = \lambda\vec{v}.$$

The eigenvalues λ of A are found by solving the following characteristic equation for λ:

$$\det(A - \lambda I) = 0.$$

Definition 53 (Index of a Nondegenerate Critical Point). Suppose p is a nondegenerate critical point of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$. The index of p is the number of negative eigenvalues of the Hessian of f evaluated at p.

Example 54. Consider the function $f(x,y) = x^2 + \cos(2y)$ on $M = \mathbb{R}^2$.

We know from before that $(0,0)$ is a critical point of f on M.

Computing the Hessian we have that

$$H_f(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & -4\cos(2y) \end{bmatrix}$$

We know then at our critical point $(0,0)$ that

$$H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$$

Therefore, we see there is 1 negative eigenvalue of $H_f(p)$. Therefore, the index of p is 1.

3.2 Morse Functions

Now that we have a thorough understanding of critical points, we move into defining a Morse function on a space.

Definition 55 (Morse function). [9] A smooth function $f : M \to \mathbb{R}$ defined on a subset of points $M \subseteq \mathbb{R}^n$ is said to be a Morse function if all of its critical points are nondegenerate.

Example 56. Note that from last section the function $f(x,y) = x^2 + \cos(2y)$ on $M = \mathbb{R}^2$ is a Morse function since its critical points are nondegenerate.

Example 57. We use angles θ to indicate points on the circle S^1, where θ and $\theta + 2\pi$ correspond to the same point on S^1. Define a function $f : S^1 \times S^1 \to \mathbb{R}$ on the torus $S^1 \times S^1$ by $f(\theta,\phi) = (R + r\cos\phi)\cos\theta$, where R and r are positive constants with $R > r$. We will show this function is a Morse function on the torus.

We see, $\frac{\partial f}{\partial \theta} = -(R + r\cos\phi)\cos\theta$, and $\frac{\partial f}{\partial \phi} = -r\sin\phi\cos\theta$. Which gives us the critical points $(0,0), (0,\pi), (\pi,0)$ and (π,π). It is clear that the Hessian at each of these points is nonzero, and therefore the critical points are nondegenerate. Therefore, $f(\theta,\phi)$ is a Morse function. Also, we see that $(0,0)$ has index 0, $(0,\pi)$ and (π,π) have index 1, and finally $(\pi,0)$ has index 2.

In this example we see that T^2 has four critical points of indices $0,1,1,2$. Another way of seeing this is through the height function $f(x,y,z) = z$ on T^2. Using the height function and parameterizing the function leads to the same critical points of the same index. In the image below we have the four critical points labeled, where they have index value $2,1,1,0$ from top to bottom.

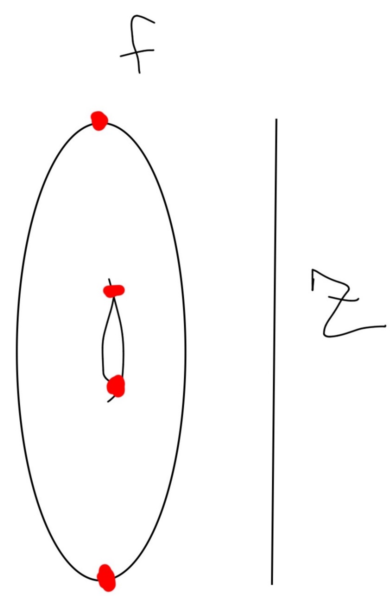

Figure 4: Height function with critical points on the torus.

Example 58. [8] Let S^3 be the unit sphere in $\mathbb{R}^3$. That is

$S^3 = \{(x,y,z,) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$. Now let $f(x,y,z) = z$ be the height function as above. We again see geometrically that the height function of a sphere would have two critical points, that of the north and south pole. We can parametrize our function to show that indeed the north and south pole of the sphere are critical points of the function f

defined on S^3. In particular we note that the index of the critical point at the north pole is 2 while the index of the critical point at the south pole is 0.

We have shown examples of Morse functions on surfaces, but the following results will help us understand how useful Morse theory is on higher dimension manifolds. We note that the following results are stated without proof. We begin with the Morse Lemma; it states that near each critical point a Morse function may be transformed into a function that is well-behaved and well-understood.

Theorem 59 (**The Morse Lemma).** *[8] Suppose M is a collection of points in $\mathbb{R}^n$ and $f : M \to \mathbb{R}$ is a Morse function defined on M. Then at each nondegenerate critical point p of f, there is a neighborhood U of p so that at each point $\vec{x} = (x_1, x_2, \ldots, x_n)$ of U, f has the form*

$$f(\vec{x}) = -x_1^2 - x_2^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_n^2,$$

where j is the index of the critical point p.

In other words, near its critical points, a Morse function closely resembles a well-behaved function. At any critical point of a Morse function, the index dictates precisely what the set of points near that critical point looks like. Therefore, we begin to see the necessity of the index of a critical point.

The following corollary helps us see that critical points on a Morse function dictate global information as a whole.

Corollary 60. *[9] Critical points of any Morse function are isolated.*

3.3 Results of Morse Theory

Up to now we have focused on results that help us determine if we have a Morse function or not. We will now move into revealing how a Morse function provides a handlebody decomposition of a space. This will be done by relating the indices of critical points of a Morse function defined on M to the cellular decomposition of M. In order to do this we must have the following notation in mind.

Suppose $f : M \to \mathbb{R}$ is a Morse function. Given a value t in the image of f, define the set

$$M_t = \{p \in M : f(p) \leq t\}.$$

We will now examine what happens to M_t as t increases.

Theorem 61. *[8] If f has no critical values in the closed interval $[a,b]$, then M_a is homeomorphic to M_b.*

This theorem tells us that the general shape of M does not change as t increases through non-critical values of f. Therefore, any changes to the topology of M must correspond to the critical points of f.

Definition 62 (**n-cell of Index λ**, or **λ-Handle**)**.** [9] For any positive integers $n \geq \lambda$, an n-cell of index λ is homeomorphic to the product of the λ-disk and the $(n-\lambda)$-disk, $D^\lambda \times D^{n-\lambda}$, where

$$D^\lambda \cong D^\lambda \times \vec{0} = \{x_1, x_2, \ldots, x_\lambda, 0, \cdots, 0) : x_1^2 + \cdots + x_\lambda^2 \leq \varepsilon\}$$

$$D^{n-\lambda} \cong \vec{0} \times D^{n-\lambda} = \{0, \ldots, 0, x_{\lambda+1}, x_{\lambda+2}, \ldots, x_n) : x_{\lambda+1}^2 + \cdots + x_n^2 \leq \delta \ll \varepsilon\}$$

The next theorem tells us precisely how each nondegenerate critical point of $f : M \to \mathbb{R}$ affects the cellular decomposition of M.

Theorem 63. *[9] Suppose $f : M \to \mathbb{R}$ is a Morse function defined on an n-dimensional space. Suppose also that c is a critical point of f with index λ. Then there is an $\varepsilon > 0$ with $M_{c+\varepsilon}$ homeomorphic to the space obtained by attaching an n-cell of index λ to $M_{c-\varepsilon}$.*

This theorem tells us that if we know all nondegenerate critical points, and their indices of $f : M \to \mathbb{R}$, then we have a complete cellular decomposition of M. Therefore, for any manifold we can find the handlebody decomposition of the space.

Theorem 64 (Handle decomposition)**.** *[9] Any Morse function on a closed manifold M defines a cellular decomposition of M. Each handle corresponds to a different critical point, and the index of each critical point dictates the index of the handle.*

Example 65. Recall that we found a Morse function on S^2 whose critical points are of index 2 and 0. By Theorem 64 we have that a sphere can be decomposed into a 2-handle and 0-handle.

Example 66. Recall that we found a Morse function on the Torus whose critical points are of index 2, 1, 1, 0. By Theorem 64 we have that a torus can be decomposed into a 2-handle, two 1-handles and a 0-handle.

This decomposition will be directly used to help us prove a result in surgery theory. However, before we move into surgery theory it is natural to define discrete Morse theory first. In the next chapter we will discuss discrete Morse theory while paying close attention to the parallels to smooth Morse theory.

CHAPTER 4

Discrete Morse Theory

In the 1990's Robin Forman carefully extended smooth Morse theory to construct discrete Morse theory. In this chapter, we examine his extension to discrete Morse theory. While smooth Morse theory relates the cellular decomposition of a space to the critical points of a smooth function defined on the space, discrete Morse theory accomplishes the same goal using discrete functions defined on the individual cells of the decomposition.

The major difference between these theories is found in their uses of equivalence. When working in smooth Morse theory we rely heavily on the use of homeomorphisms, while in discrete Morse theory we rely heavily on the use of homotopy equivalences.

We begin this chapter by defining what is a discrete Morse function on a cellular complex. We will then show how we can define critical points on a cellular complex, and then state some classical results in discrete Morse theory.

4.1 Discrete Morse Function and Critical Points

Definition 67 (Discrete Morse function[3]). The "discrete function" that assigns numerical values to the individual cells of M (where M is a complex) is a discrete Morse function on M if the following conditions are held for each p-cell $\sigma^p \subset M$.

1. $\#\{\tau^{p+1} \succ \sigma^p : f(\tau) \leq f(\sigma)\} \leq 1$, and

2. $\#\{\rho^{p-1} \prec \sigma^p : f(\rho) \geq f(\sigma)\} \leq 1$.

This discrete function on the cells of M is a discrete Morse function if the valuess assigned to each cell follow the rule: faces of a cell have smaller function values, with at most one exception allowed for each cell.

Let us consider the following examples:

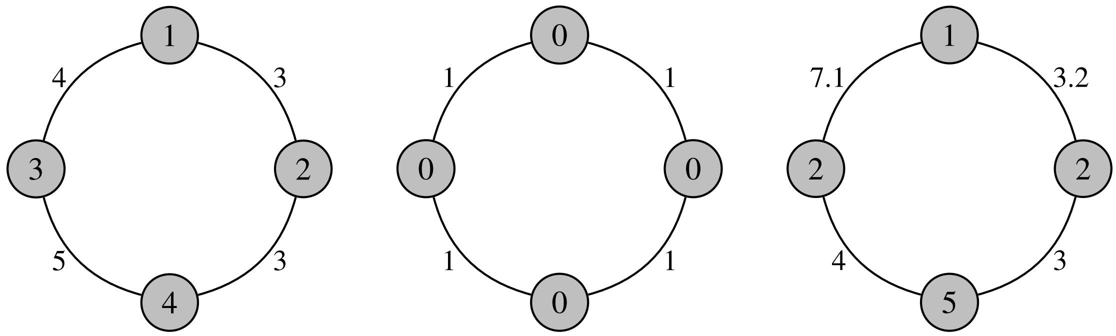

Figure 5: Discrete functions on simple complexes.

Example 68. Consider the three complexes in Figure 5 constructed by four 0-cells and four 1-cells. Each complex has a "discrete function" defined on its cells, where the function assigns numerical values to the individual cells.

We then see that the first two functions on the complexes are discrete Morse functions while the third function is not. The third function is not a discrete Morse function since the 0-cell with weight 5. The 0-cell is a face of two 1-cells who have smaller weights than 5 in particular 3 and 4. (Examples from [5])

Now just like with smooth Morse theory we must have a notion of a critical point. That is the following :

Definition 69 (Critical and Regular Points of a Discrete Morse Function [5]). Suppose $f : M \to \mathbb{R}$ is a discrete Morse function. The p-cell σ^p is a critical point of f if the following conditions are held:

1. $\#\{\tau^{p+1} \succ \sigma^p : f(\tau) \leq f(\sigma)\} = 0$, and
2. $\#\{\rho^{p-1} \prec \sigma^p : f(\tau) \geq f(\sigma)\} = 0$.

A cell that is not critical is said to be regular.

A cell in a discrete Morse function is critical if the numbers assigned to the cell follow the rule: faces of a cell have smaller function values, with no exceptions allowed for each cell.

From Example 68, we see that for the first complex every cell is critical except for the 0-cell with weight 4 and the 1-cell with weight 3. While in the second complex, every cell is critical. We do not consider the third complex since we know the function defined on the complex is not a discrete Morse function.

Now that we know what it means to have a discrete Morse function on a complex, and what a critical cell is in regards to a discrete Morse function, we can define some of the main results of discrete Morse theory.

4.2 Results from Discrete Morse Theory

We will now state the first main result of discrete Morse theory. In order to do this we must model the smooth case and observe the following notation. For any cellular complex X, suppose f is a discrete Morse function defined on X. For any real number c, define the level subcomplex X_c to be

$$X_c := \bigcup_{f(\alpha)\leq c} \bigcup_{\beta \preceq \alpha} \beta.$$

That is, the level subcomplex X_c is the set of all cells together with their faces, whose function values are no more than c. We will use this definition in the following two lemmas which lead to the main result of discrete Morse theory.

Theorem 70. *[3] If there are no critical values of the discrete Morse function $f : X \to \mathbb{R}$ on the interval $[a,b]$, then X_a is homotopy equivalent to X_b.*

Again we notice the parallel to the smooth case but instead of a homeomorphism we have a homotopy equivalence.

Theorem 71. *[3] Let X be a complex with discrete Morse function $f : X \to \mathbb{R}$. If there is a single critical cell α^k of with $f(\alpha) \in (a,b]$, then X_b is homotopy equivalent to the space obtained by attaching a k-cell e^k to X_a. That is, $X_b = X_a \cup e^k$.*

Applying these theorems to each of the critical points of a discrete Morse function, we obtain the main theorem of discrete Morse Theory.

Theorem 72. *[3] Let X be a cellular complex with discrete Morse function $f : X \to \mathbb{R}$. Then X is homotopy equivalent to the cellular complex with decomposition consisting of one n-cell for each critical cell of dimension n.*

Again we notice that there are parallels between smooth and discrete Morse theory. Most notably they both have the same form of main result.

One of the easiest ways we can work with discrete Morse functions is defining a vector field on our complex. These vector fields will simulate cellular collapse.

4.3 Discrete Vector Fields

Definition 73 (Discrete Vector Field [4]). A discrete vector field V on a complex X is a collection of pairs of cells $\{(\alpha^p, \beta^{p+1})\}$ such that $\alpha \prec \beta$ and no cell in X belongs to more than one pair in V. We often write $V(\alpha) = \beta$ to signify that the pair $(\alpha, \beta) \in V$.

When we define a discrete vector field on a complex, the vectors tell us how to perform cellular collapse.

Example 74. Figure 6 illustrates a simple vector field on a cellular complex constructed using six 0-cells, seven 1-cells and one 2-cell. Notice how the vector path indicates cellular collapse.

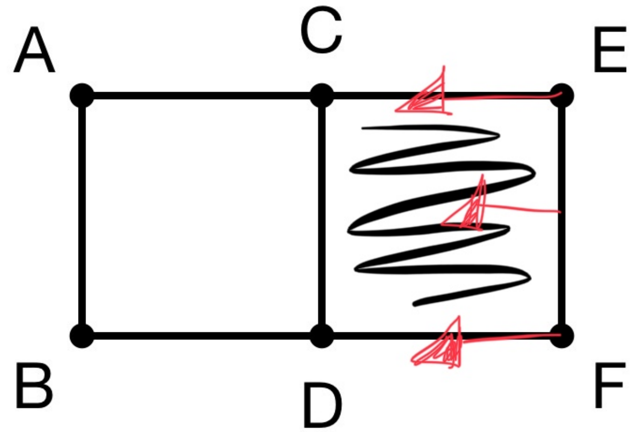

Figure 6: A simple example of a vector path.

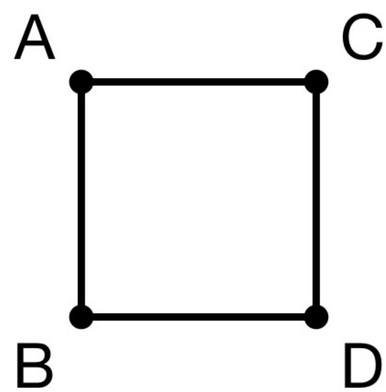

Figure 7: Result of the vector path above.

Thus we see that vector paths of a Morse function show the “flow” or how the cells collapse on each other giving a homotopy equivalence between complexes. With this definition, we immediately get the following definition and theorem.

Definition 75 (**Non-trivial Closed V-paths [4]).** Let V be a discrete vector field defined on the cells of a complex X. A V-path is a sequence of cells

$$\alpha_0^p,\ \beta_0^{p+1},\ \alpha_1^p,\ \beta_1^{p+1},\ \ldots,\ \alpha_r^p,\ \beta_r^{p+1},\ \alpha_{r+1}^p$$

such that for each $i = 0, 1, \ldots r$ we have $(\alpha_i^p, \beta_i^{p+1}) \in V$ and $\beta_i^{p+1} \succ \alpha_{i+1}^p \neq \alpha_i^p$. A V-path is said to be nontrivial and closed if $r \geq 0$ and $\alpha_0^p = \alpha_{r+1}^p$.

Theorem 76. *A discrete vector field V is associated to some discrete Morse function if and only if it contains no nontrivial closed V-paths. [4]*

These vector path can become quite complex. Consider the torus constructed using seven 0-cells, twenty-three 1-cells and eighteen 2-cells as pictured in Figure 8 together with the discrete vector path in Example 77. The vector path shows us our critical points as well as the cellular decomposition of the torus which as we know is a 0-cell, two 1-cells and a 2-cell.

Example 77. [8] In Figure 8 we see is a cellular decomposition of the torus with opposite edges of the space identified, together with a discrete vector field V. First notice that there are no closed vector paths on the torus. Therefore by Theorem 76 we know the resulting vector field identifies a discrete Morse function on our complex. Also notice any cell that does not have a vector emanating from it or into it is considered critical in our complex. In this particular example we have one critical 0-cell, two critical 1-cells, and one critical 2-cell. Thus, we know by Theorem 72 the torus represented in Figure 8 has cellular decomposition $e^0 \cup e^1 \cup e^1 \cup e^2$ which is what we expected.

Now that we have a strong foundation in both smooth and discrete Morse theory we switch our attention to surgery theory. First we will analyze surgery on embedded spheres which we will connect to cobordisms through a theorem that has a proof that relies on smooth Morse theory. This will lead us to believe that a similar theorem exists in the more general case using discrete Morse theory.

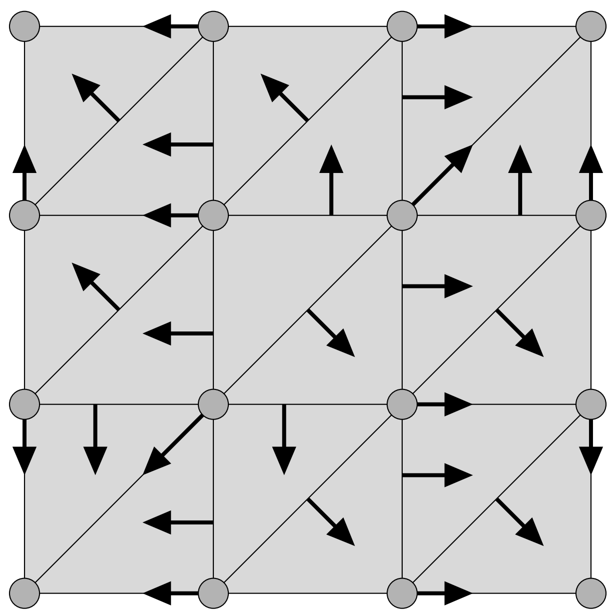

Figure 8: A vector field on a torus.

CHAPTER 5

Surgery Theory on Manifolds

5.1 Construction of Surgery Theory on Manifolds

Surgery theory on manifolds is a process in which pieces of a manifold are removed and replaced with pieces of another space, gluing the pieces along the boundary.

How do we decide what to remove and what to glue back? In order to do this we will locate spheres embedded in our manifold. We will then make clever use of the boundary operator to ensure that what is removed shares the same boundary of what is to be attached.

Consider an n-dimensional sphere S^n, which is the boundary of a disk of dimension $n+1$ is a n-dimensional sphere. That is $S^n = \partial(D^{n+1})$. However we also note that for $k \in \mathbb{N}$, $k \leq n$ then $D^{k+1} \cong D^k \times D^{n-k+1}$ therefore $\partial(D^{n+1}) = \partial(D^k \times D^{n-k+1})$.

Applying the boundary operator above we have

$$\partial(D^{n+1}) = \partial(D^k \times D^{n-k+1}) = (S^{k-1} \times D^{n-k+1}) \cup (D^k \times S^{n-k}).$$

From this information we can describe the process of surgery on a manifold as follows:

1. If S^{k-1} is embedded in M^n then there exists a neighborhood $U \subset M^n$ where $U \cong S^{k-1} \times D^{n-k+1}$. Thus we identify $(S^{k-1} \times D^{n-k+1})$ on our manifold M^n and remove it.

2. Due to our choice of U the boundary of $M^n - U$ is exactly $\partial(S^{k-1} \times D^{n-k+1})$.

3. Since the boundary of $(D^k \times S^{n-k})$ is the same as $(S^{k-1} \times D^{n-k+1})$ we may attach $(D^k \times S^{n-k})$ along the boundary of $M^n - U$ via the identity map on $(S^{k-1} \times S^{n-k})$.

This process creates a new manifold M' that is also of dimension n, where M' can be defined with the following equation [2] :

$$M' = (M^n - (S^{k-1} \times D^{n-k+1})) \cup (D^k \times S^{n-k}).$$

Note that we union via the identity map on $(S^{k-1} \times S^{n-k})$.

5.2 Examples

In order to better understand how this process works let us consider the following examples:

Example 78. Let $M^n = S^2$ with $k = 1$. Since $M^n = S^2$ we know that $n = 2$. Plugging in our values to the above equation we have

$$M' = (S^2 - (S^0 \times D^2)) \cup (D^1 \times S^1).$$

First remove $(S^0 \times D^2)$ which is two copies of disks from M^n, and thus the boundary we are left with is a pair of circles. We then attach $(D^1 \times S^1)$ or a cylinder, whose boundary is also pair of circles. This in effect attaches a 1-handle to S^2. In Figure 9 we notice that M' resembles a kettle ball from the gym, which is homeomorphic to a torus.

Example 79. Let $M^n = T^2$ with $k = 1$. Since $M^n = T^2$ we know that $n = 2$. Plugging in our values to the above equation we have

$$M' = (T^2 - (S^0 \times D^2)) \cup (D^1 \times S^1).$$

First remove $(S^0 \times D^2)$ which is two copies of disks, and thus the boundary we are

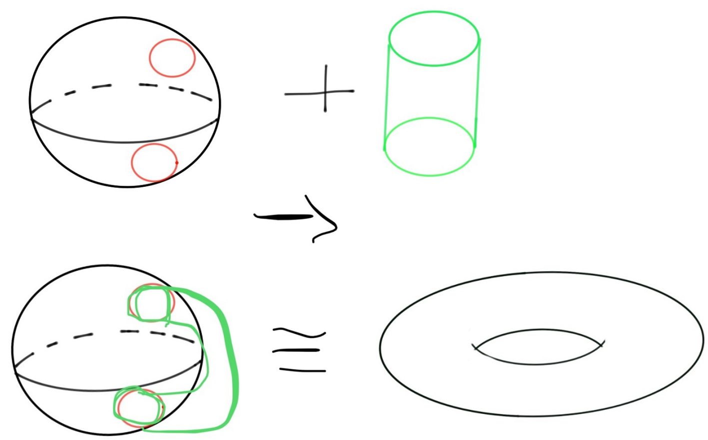

Figure 9: An example of surgery on a sphere.

left with is a pair of circles. We then attach $(D^1 \times S^1)$ or a cylinder, whose boundary is also pair of circles. This in effect attaches a 1-handle to T^2. In Figure 10 we notice that M' is homeomorphic to a genus two surface.

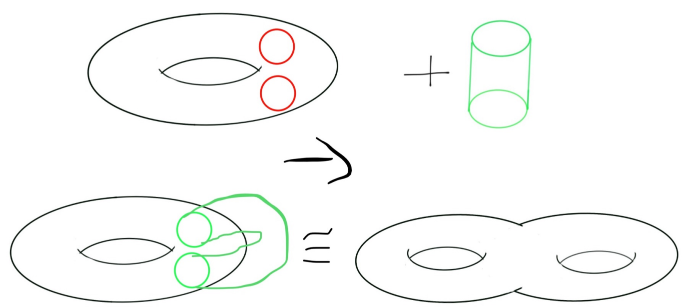

Figure 10: An example of surgery on a torus.

Example 80. Let $M^n = S^2$ with $k = 2$. Since $M^n = S^2$ we know that $n = 2$. Plugging in our values to the above equation we have

$$M' = (S^2 - (S^1 \times D^1)) \cup (D^2 \times S^0).$$

First remove $(S^1 \times D^1)$ which is a small collar on the surface. After removing $(S^1 \times$

D^1) the boundary we are left with is a pair of circles. We then attach $(D^2 \times S^0)$ or a pair of disks whose boundary is also pair of circles. In this process w are locating an embedded 1-sphere in S^2. This in effect removes a handle to S^2. In Figure 11 we notice that M' is the upper and lower hemispheres of S^2 which is homeomorphic to a pair of spheres, or $S^2 \sqcup S^2$.

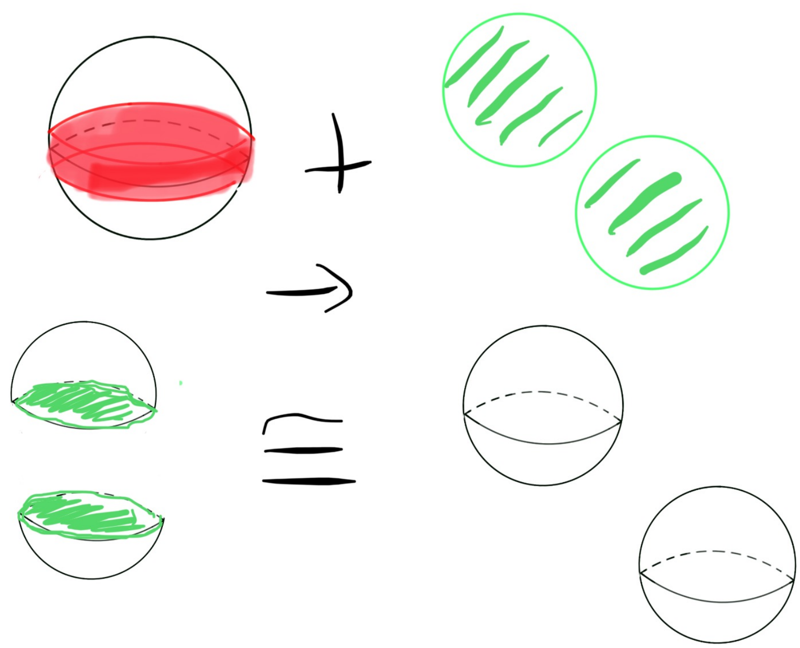

Figure 11: Removing a handle on a sphere.

Example 81. Let $M^n = T^2$ with $k = 2$. Since $M^n = T^2$ we know that $n = 2$. Plugging in our values to the above equation we have

$$M' = (T^2 - (S^1 \times D^1)) \cup (D^2 \times S^0).$$

First remove $(S^1 \times D^1)$ which is a small collar on the surface. After removing $(S^1 \times D^1)$ the boundary we are left with is a pair of circles. We then attach $(D^2 \times S^0)$ or a pair of disks whose boundary is also pair of circles. In this process we are locating an embedded 1-sphere in T^2. This in effect removes a handle of T^2. In Figure 12 we notice that M' is the torus with a portion removed, or a hollow cylinder. We know a hollow cylinder is homeomorphic to S^2.

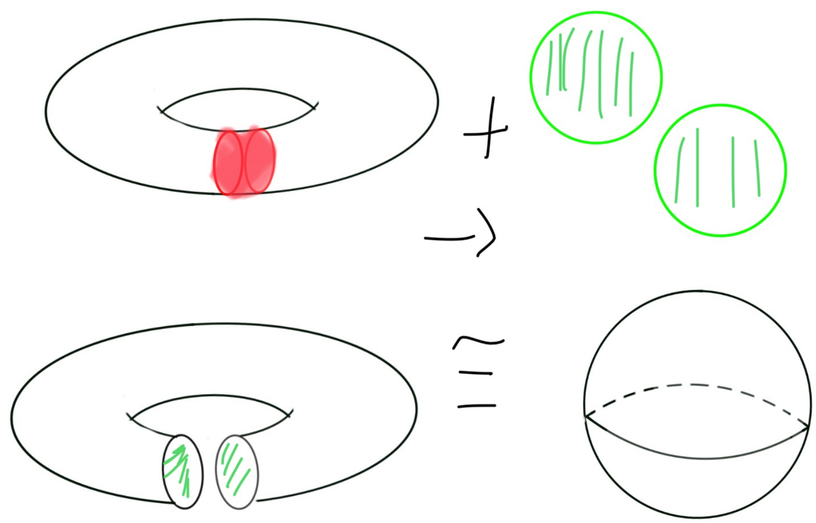

Figure 12: Removing a handle on a torus.

Surgery is much more difficult in higher dimensions. However we notice that ,in any dimension surgery drastically changes the homeomorphism type of the original manifold. Because the surgery we perform is controlled, some information is preserved.

5.3 Cobordism

Now that we have a grasp on surgery theory we will examine corbordism, which will be the tool used with surgery theory to classify manifolds of the same dimension.

Definition 82 (Cobordant). Let M and N be n -dimensional manifolds. We say M and N are cobordant if there is a compact manifold W of degree $n+1$, whose boundary is the disjoint union of M, and N i.e. $\partial(W) = M \sqcup N$. We say W is a cobordism between M and N.

It is immediately clear that for given n the notion of cobordism is an equivalence relation, dividing the set of n-manifolds into cobordism classes. Because the relation of cobordism is much weaker than that of homeomorphism, the cobordism classes of n-manifolds are much larger than homeomorphism classes of n-manifolds.

Example 83. Consider the simple example of two manifolds of degree 0, that is let $M = \{0\}$ and $N = \{1\}$. Now consider the degree 1 manifold $W = [0,1]$. Notice that $\partial(W) = \{0\} \sqcup \{1\}$, thus W is a cobordism between M and N.

Example 84. A less trivial example would be between a pair of circles and just one copy of a circle. That is let $M = S^1 \sqcup S^1$, while $N = S^1$. The cobordism W would look like a "pair of pants", as pictured in Figure 13. We note how $\partial(W)$ is exactly $N \sqcup M$. Thus we would say a pair of circles is cobordant to a single circle.

Example 85. We can find a degree 3 manifold whose boundary is the disjoint union of a torus and a sphere. That is, consider a solid $D^2 \times S^1$ where you delete an open ball or D^3, from the interior of $D^2 \times S^1$. Thus we know there exists a cobordism between a torus and a sphere, and we noticed before that we could obtain one from the other through surgery.

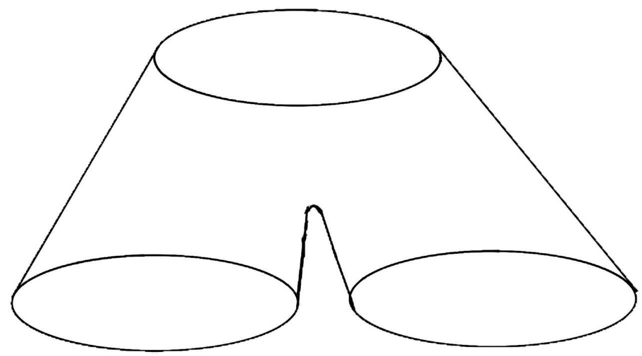

Figure 13: Cobordism W between a circle and a pair of circles.

Example 85 presents a connection between surgery and cobordism. There is an explicit connection between cobordism and surgery on embedded spheres; Morse theory provides that connection.

5.4 The link between surgery and cobordisms

Theorem 86 ([2]). *Let M and N be manifolds of the same dimension. Then M and N are cobordant if and only if M and N can be obtained from one other through a sequence of surgeries.*

We will not rigorously prove this result, but rather sketch the main idea of the proof in order to show how Morse theory is used.

Sketch of Proof

Suppose M and N are two n-dimensional cobordant manifolds. That is, there exists a $(n+1)$-dimensional manifold W, where $\partial(W) = M \sqcup N$. Through Morse theory we can identify a Morse function f on W that identifies the critical points of W. Thus using this information we can find the handlebody decomposition of W. Through this decomposition we can analyze what handles were added, or removed to give the cobordism W. Therefore this decomposition will show us the surgeries that were involved to obtain M from N or vice versa.

Conversely if we can show that two n-dimensional manifolds M and N can be obtained from one another through a series of surgeries then the surgeries give us a "blueprint" on how to create our cobordism W. That is we simply analyze what handles were deleted or added to obtain N from M or vice versa. Then this can be considered our handlebody decomposition of W and through Morse theory we can glue our handles together to obtain W.

CHAPTER 6

Cellular Surgery

The previous chapter provided a connection between surgery theory on embedded spheres in a manifold to the equivalence relation known as cobordism. This connection was due to Morse theory. The discovery of discrete Morse theory begs the question : are there similar notions of surgery and cobordism on complexes that can be connected by discrete Morse theory?

This is not an innovative idea. In 2010 Bruno Benedetti submitted a paper to the Arxiv entitled "Discrete Morse Theory Is At Least As Perfect As Morse Theory", in which he translates some applications of smooth Morse theory into the discrete setting. Currently there are no notions for surgery on a cellular complex. Therefore, we create an algorithm for performing surgery on a cellular complex. In this chapter, we define a discrete version of surgery on a complex, allowing us to make a conjecture about the discrete version of the link between surgery and cobordism like Theorem 86.

6.1 An Algorithm for Cellular Surgery

Recall that in our discussion of discrete Morse theory that vector paths play an important role. If a vector path is not closed, then the path simulates cellular collapse. Also, if a vector path is closed we are prohibited from performing discrete Morse theory. Therefore, we avoid closed vector paths on a cellular complexes. Through this, we discover that a closed vector path invokes a natural way to define surgery on a cellular complex. In order to describe our algorithm for cellular surgery we must first note the following definitions.

Definition 87 (Plain Faces). Let V be a vector path on a cellular complex X. Let $\mathscr{V}$ be the collection of cells in our vector path V, also let $\tau \in \mathscr{V}$. Then we say

$\gamma \in \mathscr{F} \iff \gamma \prec \tau$ and $\gamma \notin \mathscr{V}$. We call $\mathscr{F}$ the set of plain faces of $\mathscr{V}$.

Definition 88 (Pure Vector Path). Let V be a closed $(k,k+1)$ vector path of length m where $\tau_1, \tau_2, \ldots, \tau_{2m}$ are the cells in V, where τ_{2m+1} is a k-cell and τ_{2m} is a $(k+1)$-cell. We say a vector path is pure if for each τ_i, the only cells of co-dimension 1 that τ_i shares a face with are τ_{i-2} or τ_{i+2}. Otherwise we say the vector path is pure.

In order to visualize what we mean by a pure vector path consider the following example of a vector path that is not pure.

Example 89. In Figure 14 we have a vector path that is not pure. Notice if we consider the blue 1-cell FH the starting point of our closed $(1,2)$ vector path then the 2-cells $CEFD$ and $DFKJ$ share a face DF. However $CEFD$ is the fourth cell in our vector path, while $DFKJ$ is the tenth cell in our vector path. Therefore since they share a face but are not consecutive of each other then they are not pure vector paths.

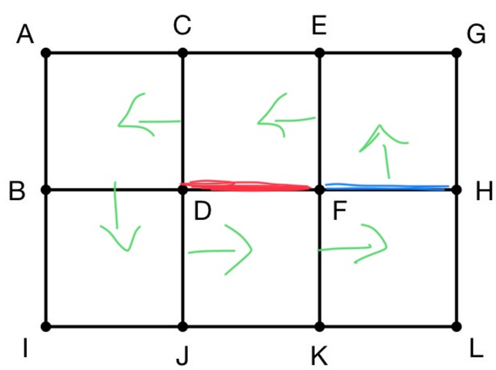

Figure 14: A not pure vector path

Now that we have the required definitions, we have the following algorithm:

Cellular Surgery

Let X be a CW-complex of dimension n. Let $k \in \mathbb{N}$ where $k < n-1$. Then :

1. Find a pure $(k, k+1)$ vector path on X.

2. Remove all cell in $\mathscr{V}$.

3. Also remove any cells who lost their face.

   That is, say σ is a k-cell that was removed, and $\tau_1, \tau_2, \ldots, \tau_p$ are $(k+1)$-cells that are not in the vector path, where σ is the face of $\tau_1, \tau_2, \ldots, \tau_p$. Then we remove $\tau_1, \tau_2, \ldots, \tau_p$ as well.

4. **Claim :** The deleted cells form $(S^1 \times D^k)$. Therefore the boundary of the $(S^1 \times D^k)$ form the cells in $\mathscr{F}$. Thus the boundary of a $(D^2 \times S^{k-1})$ would have the same boundary as the $(S^1 \times D^k)$.

5. Attach $(D^2 \times S^{k-1})$ to $\mathscr{F}$.

Proof of Claim. We recall that from smooth surgery we located and removed $(S^{k-1} \times D^{n-k+1})$ then attached $(D^k \times S^{n-k})$. We could do this since $(S^{k-1} \times D^{n-k+1})$ and $(D^k \times S^{n-k})$ shared the same boundary. We will show that by mimicking surgery on embedded spheres our claim is true.

Since we are locating a closed vector path, the path $(k, k+1)$ by definition identifies a S^1. Therefore, our $(k, k+1)$ vector path identifies an embedded 1-sphere in our complex. Thus since we are mimicking surgery on embedded spheres we identify a sub-complex of X that is homotopy equivalent to $(S^1 \times D^m)$ for some m.

However, since we are finding a $(k, k+1)$ vector path the total dimension of what we remove must be $k+1$. Therefore we see $(S^1 \times D^m)$ has total dimension $m+1$. Thus we must have $m = k$.

Now since we are mimicking smooth surgery we know since we remove $(S^1 \times D^k)$ we have that $(k-1) = 1$ and $(n-k+1) = k$. Therefore, we have that $k = 2$ and $n-k = k-1$. Thus we attach $(D^2 \times S^{k-1})$, since it shares a boundary, mainly $(S^1 \times S^{k-1})$ as $(S^1 \times D^k)$. □

We see that because of the above proof, our proposed algorithm for extending surgery to the discrete setting is consistent with notions that are established in the smooth case. Also, it is worth noting that if we have a CW-complex that is a manifold, we can perform surgery on the complex in the exact same way as we would in the smooth case. That means that our new algorithm is an extension of surgery to non-manifolds. However, we will see in a few examples that if we operate on a manifold, cellular surgery resembles smooth surgery when $n = 2$ and $k = 2$.

6.2 Examples of Discrete Surgery

Example 90. Consider the cube constructed from eight 0-cells, twelve 1-cells and six 2-cells as pictured in Figure 15.

Now choose a closed $(1,2)$-vector path on our complex, which is pictured in figure 16 in green. Since we are choosing a $(1,2)$ vector path our we are identifying and removing $(S^1 \times D^1)$ while attaching $(D^2 \times S^0)$. We know from Example 80 that applying this process to a sphere results in a space that is homeomorphic to two copies of a sphere.

Now due to our choice of vector path we have that the 1-cells DC,EF, JI, and HG are in set $\mathscr{V}$. Also we have the 2-cells $DCFE, EFIJ, JIGH, HGCD$ are in set $\mathscr{V}$.

We note due to our choice of vector path that the 0-cells D,H,J,E,I,F,C,G are in

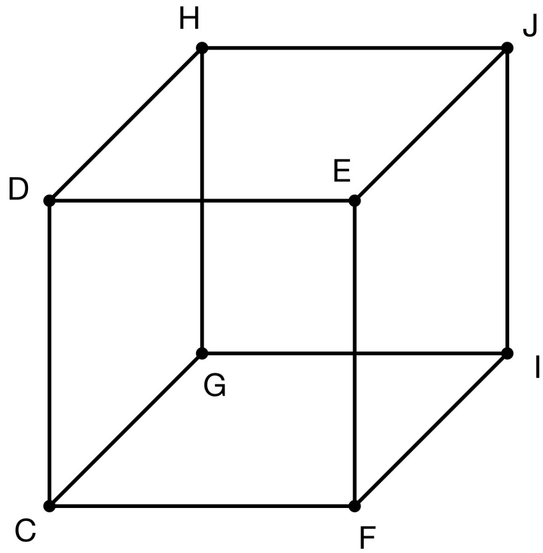

Figure 15: A cube

$\mathscr{F}$, as well as 1-cells CG, GI, IF, FC, DH, HJ, JE, ED. In Figure 16 we colored these cells blue.

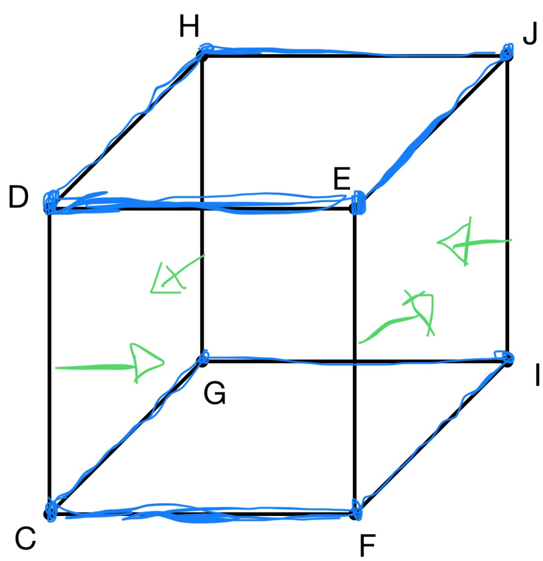

Figure 16: Vector path on a cube

Remove all cells in the set $\mathscr{V}$. Then attach $(D^2 \times S^0)$ along the blue cells. This is represented by the red bubbles in Figure 17. We notice the resulting complex is homeomorphic to two copies of spheres, as we expected.

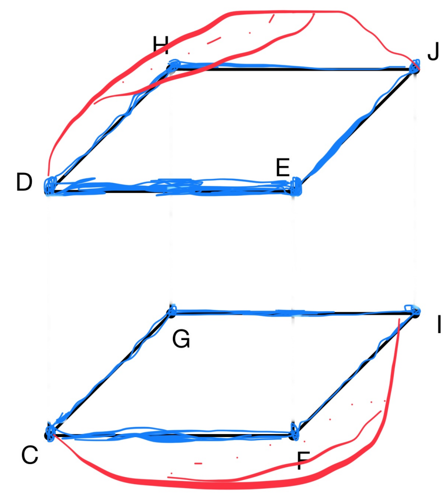

Figure 17: Resulting surgery on a cube

For the next example we will consider a complex which is homeomorphic to a torus, see Figure 18.

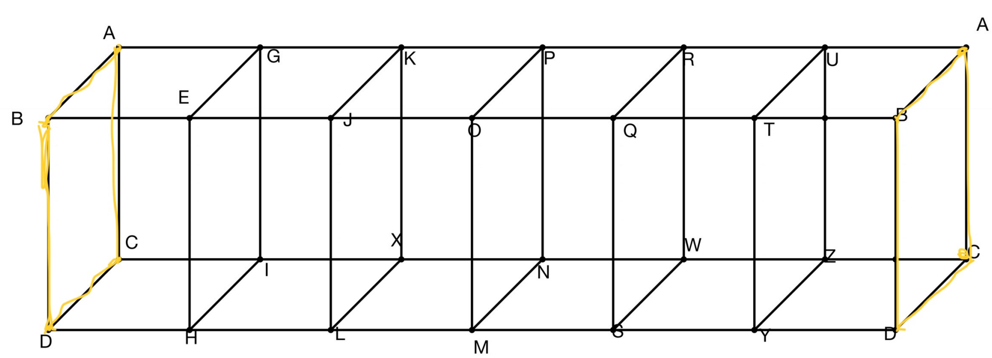

Figure 18: A subdivided torus before the yellow 1-cells are identified.

Example 91. Again we perform cellular surgery on a space that we know the outcome produced by smooth surgery. That is we perform cellular surgery on a complex that is homeomorphic to a torus.

Choose a closed $(1,2)$-vector path on our complex, which is pictured in Figure 19. Since we are choosing a $(1,2)$-vector path we are identifying and removing $(S^1 \times D^1)$ while attaching $(D^2 \times S^0)$. We know from before that applying this process to a torus results in a space that is homeomorphic to a sphere.

Now due to our choice of vector path we have that the 1-cells OQ,PR,NW, and MS are in set $\mathscr{V}$. Also we have the 2-cells $OQRP, PNWR, NWSM, MSQO$ are in set $\mathscr{V}$.

We see due to our choice of vector path that the 0-cells O,P,R,Q,W,N,M,S are in $\mathscr{F}$, as well as 1-cells OP, PN, NM, MO, SQ, QR, RW, WS. In Figure 19 we colored these cells blue.

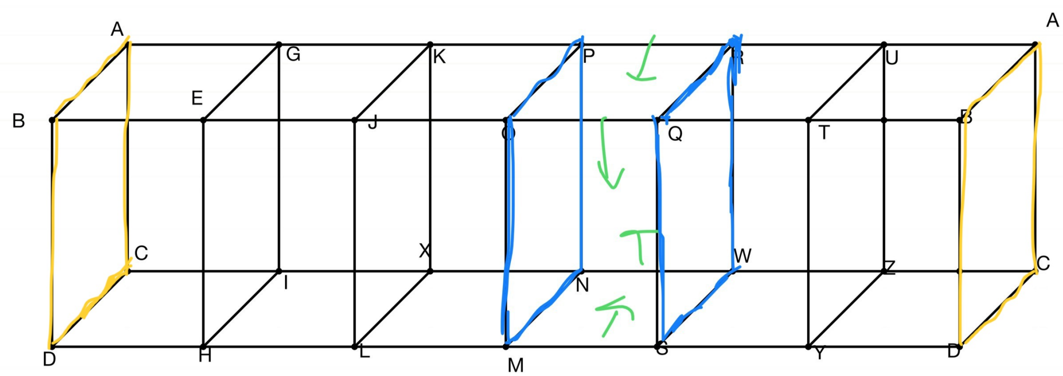

Figure 19: Our chosen vector path on our torus.

Remove all cells in the set $\mathscr{V}$. Then attach $(D^2 \times S^0)$ along the blue cells. Therefore we have the red walls in Figure 20. We notice the resulting complex is a cylinder with caps on the ends, which is homeomorphic to a sphere, which again is what we expect.

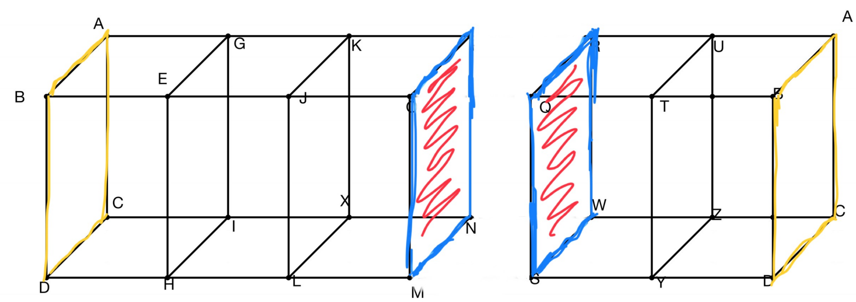

Figure 20: Resulting surgery on a torus

Now we will move into examples of cellular surgery on CW-complexes that are not manifolds. An example of a non-manifold is pictured in Figure 21 which we will call a "Torus with Rooms". Notice how there are walls that divide the torus into "rooms", in this example our "Torus with Rooms" has a total of 6 rooms.

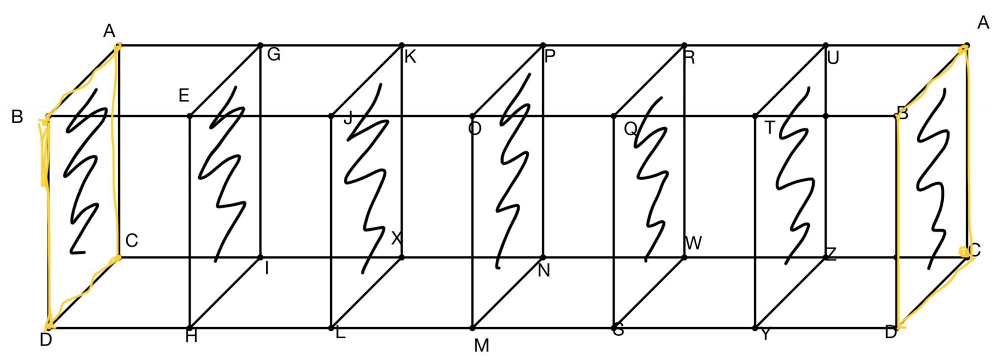

Figure 21: "A Torus with Rooms" before the yellow walls are identified.

Example 92. Choose a closed $(1,2)$-vector path which is in green in Figure 22. Since we have chosen a a closed $(1,2)$-vector path on our complex we are identifying and removing $(S^1 \times D^1)$ while attaching $(D^2 \times S^0)$.

Due to our choice of vector path we have that the 1-cells OQ,PR,NW, and MS are in set $\mathscr{V}$. Also we have the 2-cells $OQRP$,$PNWR$,$NWSM$,$MSQO$ in set $\mathscr{V}$.

Also we see due to our choice of vector path that the 0-cells O,P,R,Q,W,N,M,S are in $\mathscr{F}$, as well as 1-cells OP, PN, NM, MO, SQ, QR, RW, WS. In Figure 22 we colored these cells blue.

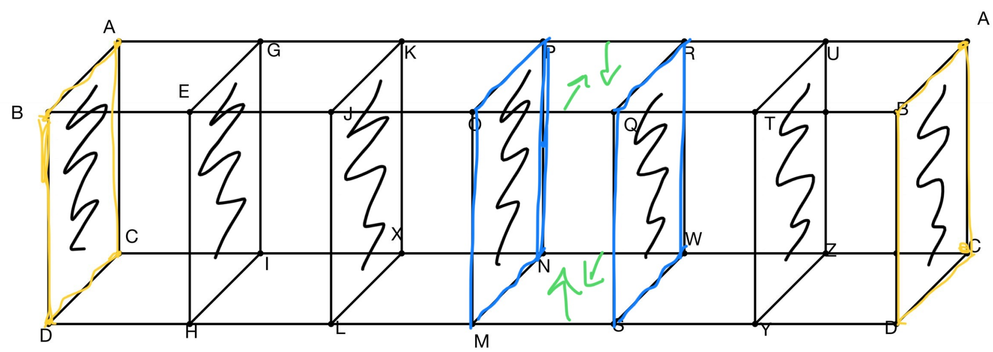

Figure 22: A closed vector path on a "Torus with Rooms"

Remove all cells in the set $\mathscr{V}$. Then attach $(D^2 \times S^0)$ along the blue cells. Therefore we have the red bubbles in Figure 23. Essentially this surgery deleted one "room" and created two new "rooms".

Example 93. Choose a different closed $(1,2)$-vector path on the "Torus with Rooms" which is in green in Figure 24 below. Note since we are choosing a $(1,2)$ vector path we are identifying and removing $(S^1 \times D^1)$ and attaching $(D^2 \times S^0)$.

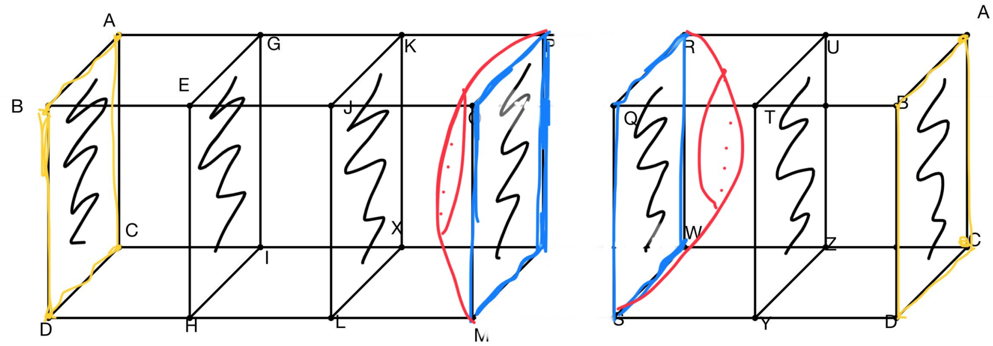

Figure 23: A surgery on a torus with rooms

Due to our choice of vector path we have that the 1-cells OM,QS,RW, and PN are in set $\mathscr{V}$. Also we have the 2-cells $PNMO, OQSM, QWR, RWNP$ are in set $\mathscr{V}$.

We see due to our choice of vector path that the 0-cells O,P,R,Q,W,N,M,S are in $\mathscr{F}$, as well as 1-cells OP, PR, RQ, QO, WS, SM, MN, NW. In Figure 24 we colored these cells blue.

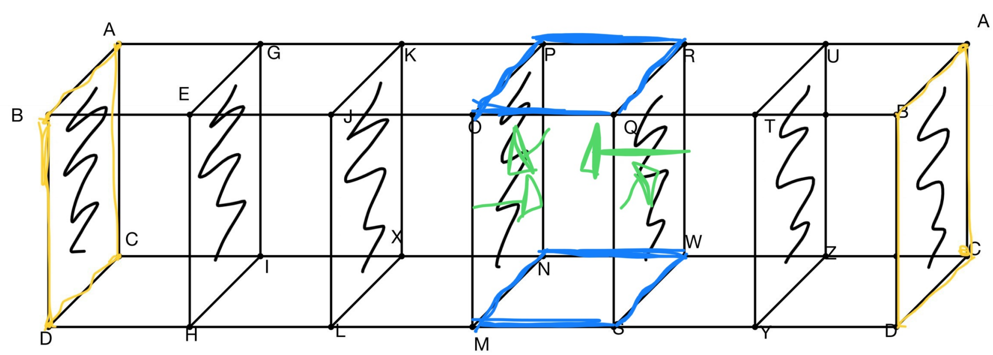

Figure 24: A different vector path on the "Torus with Rooms"

Remove all cells in the set $\mathscr{V}$. However by our algorithm, the listed 1-cells in $\mathscr{V}$ are faces of higher dimensional cells. Therefore the higher dimensional cells must be removed as well. That is the 2-cells whose faces are $JOML, KXNP, RUZW, SYTQ$ are removed. Then attach $(D^2 \times S^0)$ along the blue cells. Therefore we have the red bubbles in Figure 25. Essentially this surgery deleted three "rooms" and created two new "rooms".

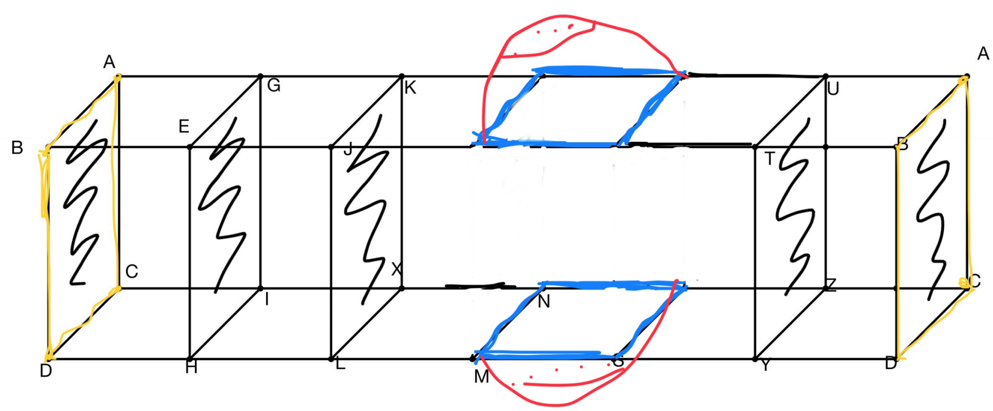

Figure 25: Another surgery on a "Torus with Rooms"

Example 94. Choose a different closed $(1,2)$-vector path on the "Torus with Rooms" which is in green on the Figure 26. Since we are choosing a $(1,2)$ vector path we are identifying and removing $(S^1 \times D^1)$ and attaching $(D^2 \times S^0)$.

By our choice of vector path we have that the 1-cells OM,QS,RW, PN, KX, and JL are in set $\mathscr{V}$. Also we have the 2-cells $JOML, OQSM, QSWR, RWNP, PKXN, KJLX$ are in set $\mathscr{V}$.

Lastly we see due to our choice of vector path that the 0-cells J,O,Q,R,P,K,L,M,S,W,N,X are in $\mathscr{F}$, as well as 1-cells JO, OQ, QR, RP, PK, KJ LM, MS, SW, WN, NX, XL. In Figure 26 we colored these cells blue.

Remove all cells in the set $\mathscr{V}$. However by our algorithm, the listed 1-cells in $\mathscr{V}$ are

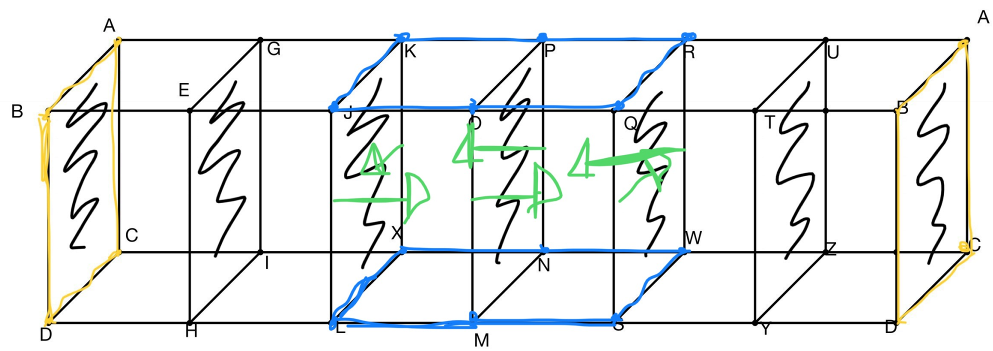

Figure 26: A vector path on a "Torus with Rooms"

faces of higher dimensional cells. Therefore the higher dimensional cells must be removed as well. That is the 2-cells $EHLJ, GKXI, RUZW, SYTQ, OPNM$ are removed. Then attach $(D^2 \times S^0)$ along the blue cells. Therefore we have the red bubbles in Figure 27. Essentially this surgery deleted four "rooms" and created two new "rooms".

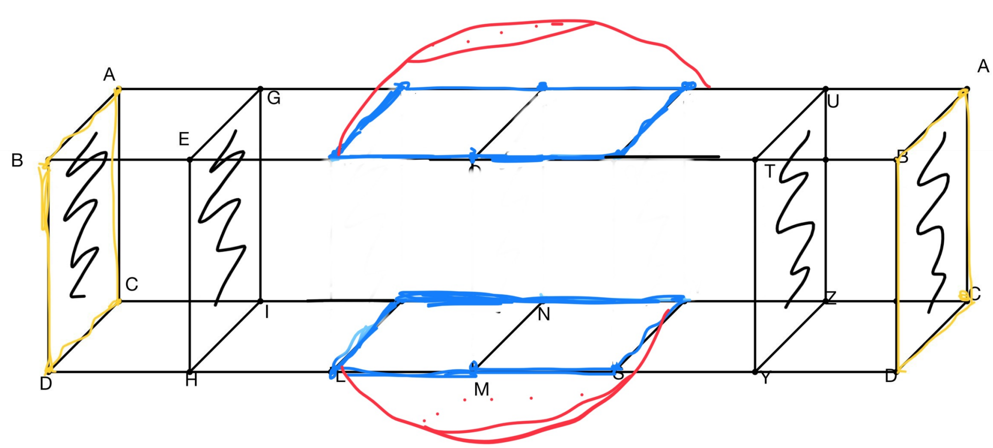

Figure 27: A different surgery on a torus with rooms

Example 95. Choose a different closed $(1,2)$-vector path on the "Torus with Rooms" which is in green on the Figure 28. This one may be harder to picture, but our path revolves on the inner circle of the "torus with rooms". Note since we are choosing a $(1,2)$ vector path we are identifying and removing $(S^1 \times D^1)$ while attaching $(D^2 \times S^0)$.

By our choice of vector path we have that the 1-cells BD, EH, JL, OM, QS, TY are in our set $\mathscr{V}$. Also we have the 2-cells $BOHE, EHLJ$, $JLMO, OMSQ$, $QSYT, TYDB$ are in set $\mathscr{V}$.

Finally we see due to our choice of vector path that the 0-cells B,E,J,O,Q,T,D,H,L,M,S,Y are in $\mathscr{F}$, as well as 1-cells BE, EJ, JO, QT, TB, DH, HL,LM,MS, ST, TD. In figure 28 we colored these cells blue.

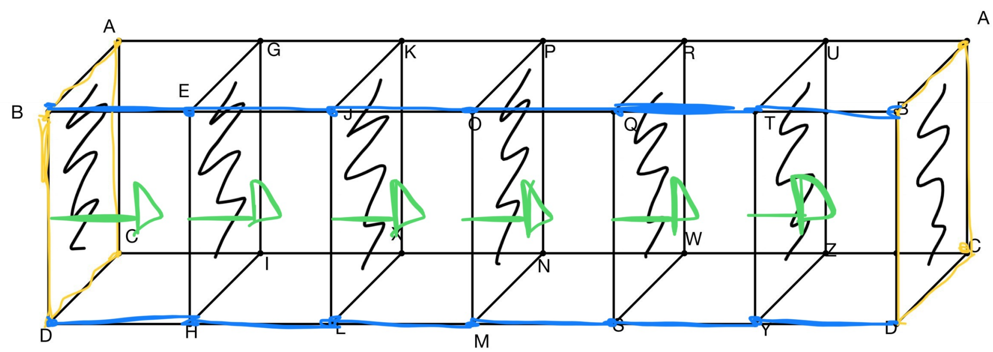

Figure 28: Another closed vector path on the "Torus with Rooms"

Remove all cells in the set $\mathscr{V}$. However by our algorithm, the listed 1-cells in $\mathscr{V}$ are faces of higher dimensional cells. Therefore the higher dimensional cells must be removed as well. That is the 2-cells $BDCA, EHIG, JKXL, OPNM, QRWS, TUZY$ are removed. Then attach $(D^2 \times S^0)$ along the blue cells. Therefore we have the red bubbles in Figure 29. Essentially this surgery deleted all "rooms" and adds one "room".

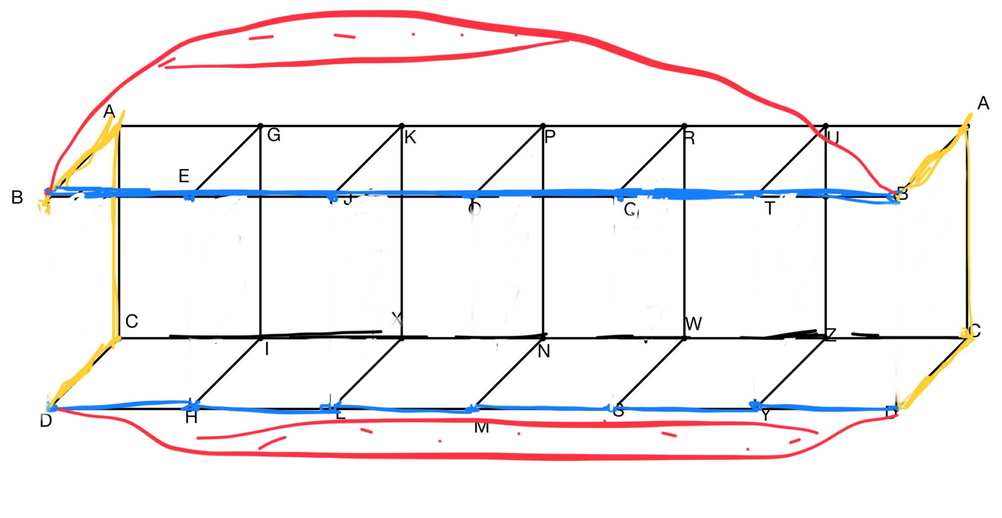

Figure 29: One more surgery on a "Torus with Rooms"

Now we consider a different CW complex that is not a manifold. This time we consider a tetrahedron and a cube sharing only one 1-cell as pictured in Figure 30.

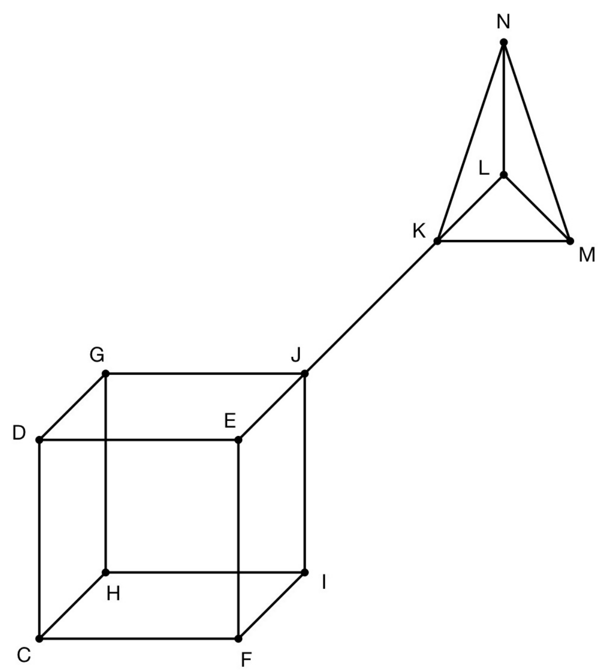

Figure 30: A cube and tetrahedron connected by a line

Example 96. Choose a closed $(1,2)$-vector path which is in green in Figure 31. Note since we are choosing a $(1,2)$-vector path we are identifying and removing $(S^1 \times D^1)$ while attaching $(D^2 \times S^0)$.

By our choice of vector path we have that the 1-cells DC,EF JI GH are in set $\mathscr{V}$. Also the 2-cells $DCHG, EFIJ, GHIJ, DCFE$ are in set $\mathscr{V}$.

Finally, we see due to our choice of vector path the 0-cells D, E, J, G, C, H, F, I are in set $\mathscr{F}$ as well as the 1-cells CH, HI, IF, FC, DE, EJ, JG, GD. In figure 31 we colored these cells blue.

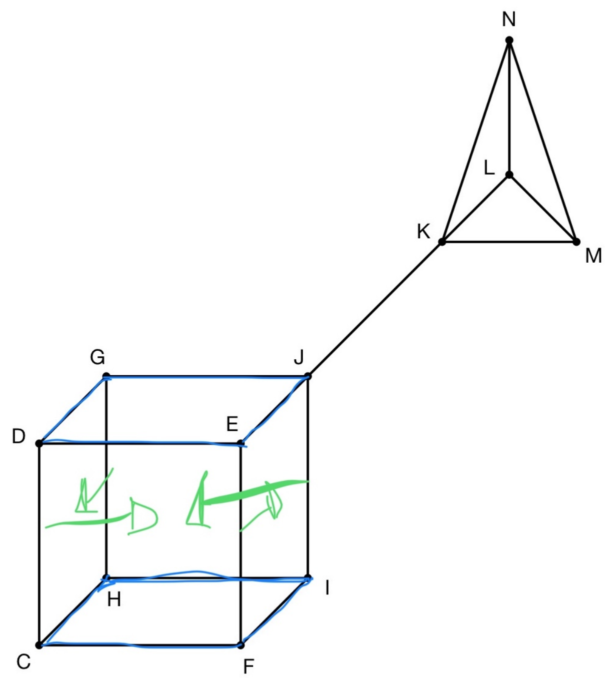

Figure 31: A vector path on a cube and tetrahedron connected by a line.

Remove all cells in the set $\mathscr{V}$. Then attach $(D^2 \times S^0)$ along the blue cells. Therefore we have the red bubbles in Figure 32. Essentially this surgery deleted one "room" and adds two "rooms".

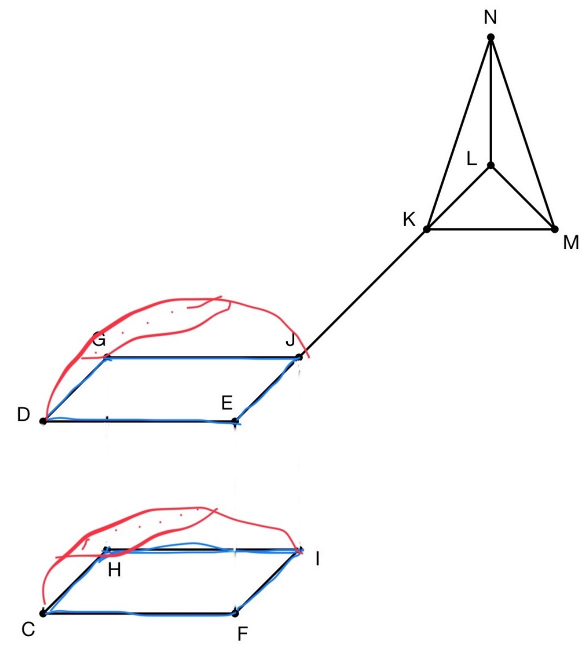

Figure 32: Surgery on a cube and tetrahedron connected by a line

Example 97. Choose a different closed $(1,2)$-vector path on a cube and tetrahedron sharing only one 1-cell which is in green in Figure 33. Note since we are choosing a $(1,2)$ vector path we are identifying and removing $(S^1 \times D^1)$ and attaching $(D^2 \times S^0)$.

By our choice of vector path we have that the 1-cells KN,MN NL are in set $\mathscr{V}$. Also the 2-cells NLK, KNM, NLM are in set $\mathscr{V}$.

Finally we see due to our choice of vector path the 0-cells N, L, K, M are in set $\mathscr{F}$ as well as the 1-cells KL, LM, MK. In figure 33 we colored these cells blue.

Remove all cells in the set $\mathscr{V}$. Then attach $(D^2 \times S^0)$ along the blue cells. Therefore we have the red bubbles in Figure 34. Essentially this surgery deleted one "room" and adds two "rooms".

We will consider one last example of a space that is not a manifold. We consider 4 tetrahedra who share exactly one point, which we call a wedge of tetrahedra, which is homeomorphic to a wedge of 4 spheres, see Figure 35.

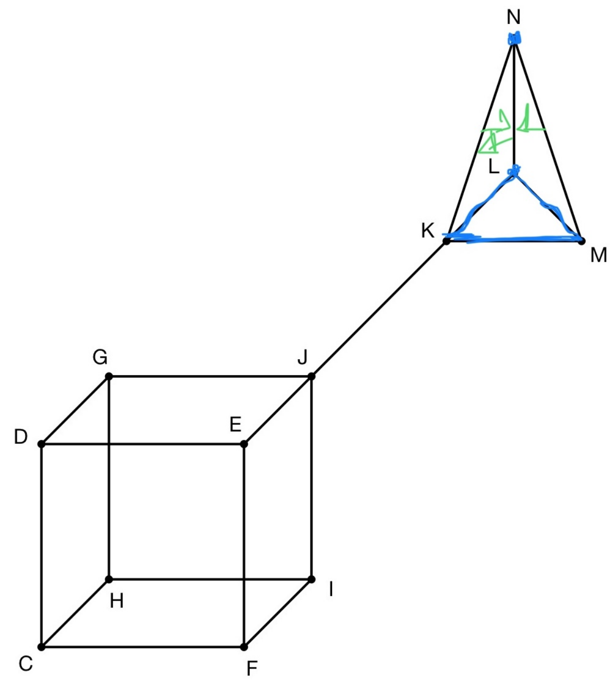

Figure 33: A different vector path on a cube and tetrahedron connected by a line.

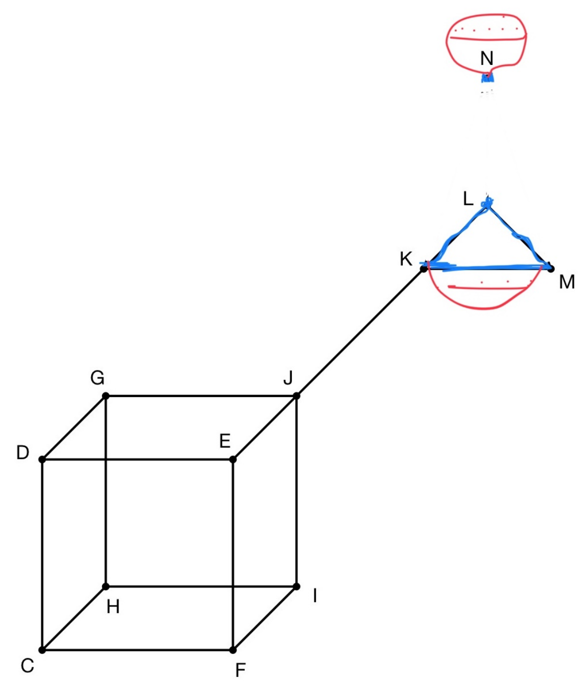

Figure 34: Different surgery on a cube and tetrahedron connected by a line.

Example 98. Choose closed $(1,2)$-vector path which is in green in Figure 36, notice no matter the choice of vector path we will have the same result. Note since we are choosing

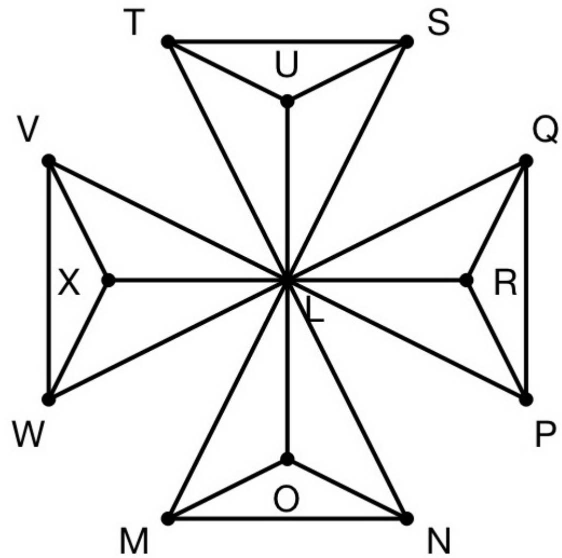

Figure 35: Wedge of tetrahedra

a $(1,2)$ vector path we are identifying and removing $(S^1 \times D^1)$ and attaching $(D^2 \times S^0)$.

By our choice of vector path we have that the 1-cells LM,LO LN are in set $\mathscr{V}$. Also the 2-cells LMO, LMN, NLO are in set $\mathscr{V}$.

Finally we see due to our choice of vector path the 0-cells L, M, N, O are in set $\mathscr{F}$ as well as the 1-cells MO, ON, NM. In figure 36 we colored these cells blue.

Remove all cells in the set $\mathscr{V}$. Then attach $(D^2 \times S^0)$ along the blue cells. Therefore we have the red bubbles in Figure 37. Essentially this surgery deleted one "room" and adds two "rooms".

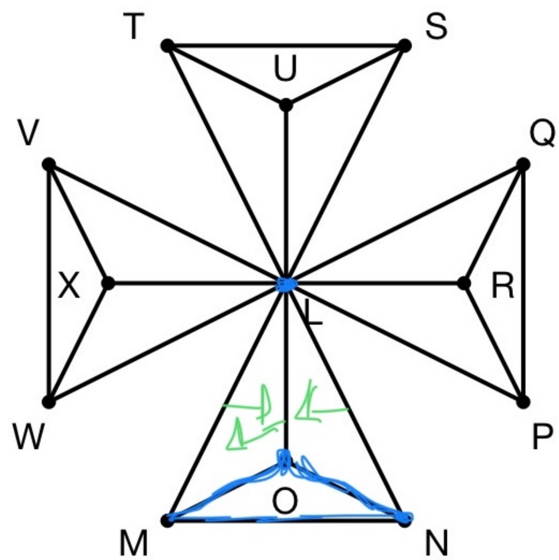

Figure 36: A closed vector path on a wedge of tetrahedra.

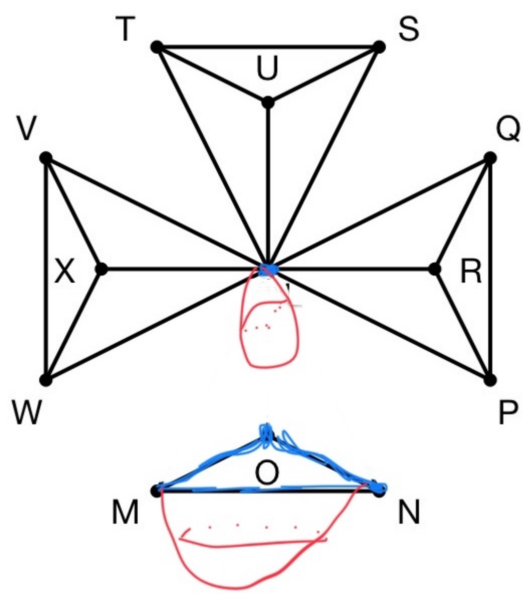

Figure 37: Surgery on a wedge of tetrahedron

CHAPTER 7

Expanding Cobordism

In the previous chapter we constructed an algorithm for generalizing surgery to cellular complexes. We recall that in the smooth case cobordisms and surgery were connected through smooth Morse theory. Therefore we require our generalization of cobordant to be connected cellular surgery. We will use the definition of cellular surgery to help define our generalized definition of cobordism.

We will begin by stating the definition for our generalization of cobordant. Then we will describe how to construct our cobordism before finally giving some examples.

7.1 Locally Cellular Cobordism

Definition 99 (Locally Cellular Cobordant). Let M,N be two CW-complexes with X and Y sub-complexes of M and N respectively. Then we say M and N are locally cellular cobordant if we can find a cellular complex $\mathscr{W}$ where X and Y are sub-complexes of $\mathscr{W}$.

We observe that in this definition we need to construct a new complex $\mathscr{W}$ where X and Y are sub-complexes. In order to construct $\mathscr{W}$ we first define a specific form of subdivision on a complex.

Definition 100 (Vector Path Subdivision). Let X be a CW-complex with a $(k,k+1)$ vector path V of length m. Let $\tau_1,\tau_2,\ldots,\tau_{2m}$ be the cells in our vector path, where τ_i is a k-cell if i is odd, and a $k+1$ cell if i is even. Let $v_j \in V$ be an individual vector in our path where v_j begins at τ_j and ends at τ_{j+1} where j is odd.

We subdivide τ_j into three components where $\tau_j = \alpha_j \cup \gamma_j \cup \beta_j$ where α_j and β_j each share a $k-1$ cell with γ_j.

We also subdivide τ_{j+1} into three components where $\tau_{j+1} = \alpha_{j+1} \cup \gamma_{j+1} \cup \beta_{j+1}$

where α_{j+1} and β_{j+1} share a k cell with γ_{j+1}.

Doing this for all $v_j \in V$ we call this a vector path subdivision and denote our resulting complex ΔX.

Consider the following examples to better visualize the subdivision.

Example 101. Consider the cellular complex made of six 0-cells seven 1-cells and two 2-cells in Figure 38 below with a $(1,2)$ vector path V in green.

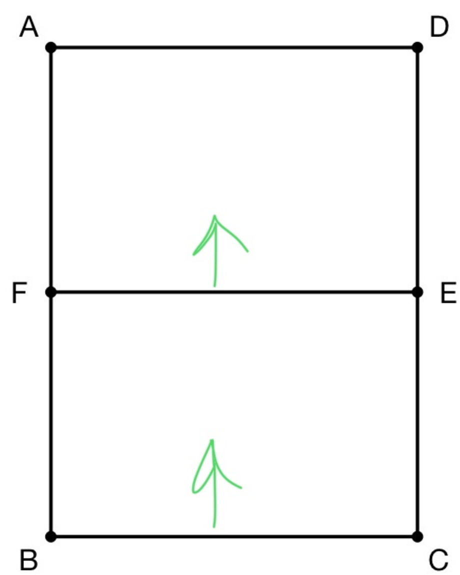

Figure 38: Cellular complex with a vector path

We then apply our vector path subdivision to our complex. We see that the 1-cell BC, becomes BG, GH, HC, and the 1-cells FE and AD become FI, IJ, JE and AK, KL, LD respectively. Also the 2-cells $FBCE$ and $ADFE$ become $FIBG, IJHG, JEHC$ and $AKIF, KLJI, LDEJ$ respectively.

Note that $\{\alpha_j : j \in \mathbb{Z} \text{ and } 1 \le j \le 5\} = BG, BGIF, FI, FIKA, AK$,

$\{\gamma_j : j \in \mathbb{Z} \text{ and } 1 \le j \le 5\} = GH, GHJI, IJ, IJKLKL$ and

$\{\beta_j : j \in \mathbb{Z} \text{ and } 1 \le j \le 5\} = HC, HCEJ, JE, JEDL, DL.$

The resulting complex is pictured in Figure 39.

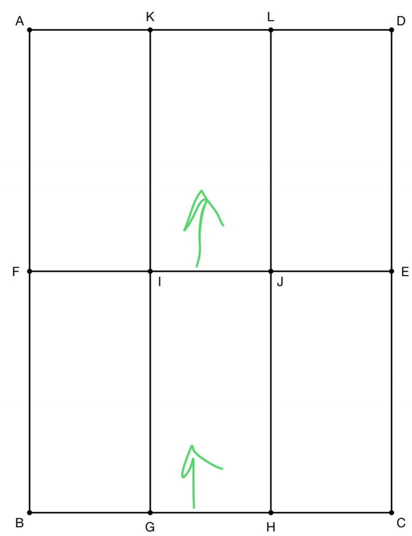

Figure 39: Cellular complex with vector path subdivision

Example 102. Consider the cube created using six 0-cells, twelve 1-cells, and six 2-cells with a $(1,2)$ vector path V pictured in Figure 40.

Applying our vector path subdivision we obtain the complex in figure 41.

Notice that in Example 102 our V was a closed vector path which we utilized in construction of our generalization of surgery to define what to remove and attach. Again we utilize closed vector paths this time in regards to creating $\mathscr{W}$. However, we require the following lemma in order to show our construction of $\mathscr{W}$ is well defined.

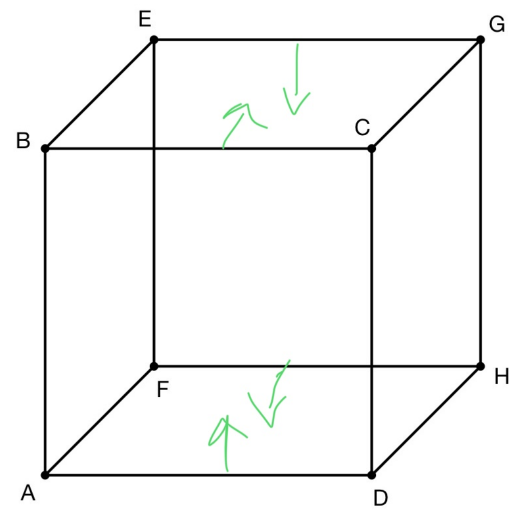

Figure 40: Cube with vector path

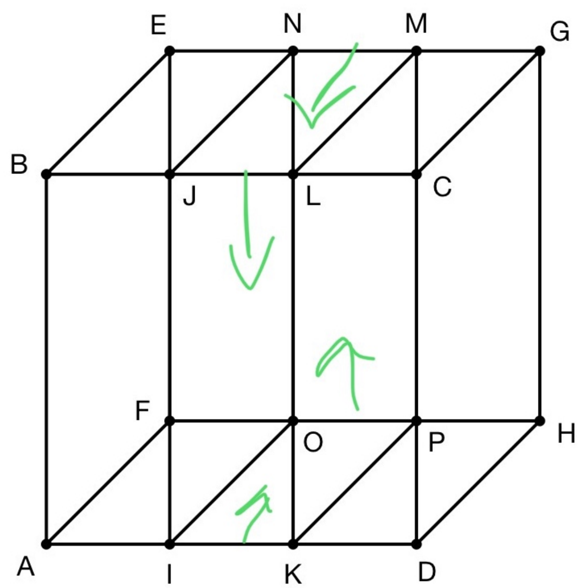

Figure 41: Cube with vector path subdivision

Lemma 103. *Let X be a CW-complex with a closed $(k, k+1)$ vector path V. Let $\dagger X$ be the vector path subdivision with respect to V. Then $\{\alpha_j\}_j$ and $\{\beta_j\}$ each form sub-complexes homeomorphic to $S^1 \times D^k$.*

Proof of Lemma. Recall that we proved when defining surgery on cellular complexes that our closed vector path locates $(S^1 \times D^k)$. Now, since we have a closed vector path V the cells in out path identify $(S^1 \times D^k)$. When subdividing our complex $\{\alpha_j\}_j, \{\gamma_j\}_j, \{\beta_j\}_j$ still follow our vector path by definition. Therefore by our definition of our generalization of surgery $\{\alpha_j\}_j, \{\gamma_j\}_j, \{\beta_j\}_j$ are homeomorphic to $(S^1 \times D^k)$. $\square$

Note that a direct consequence of lemma 104 provides that we can attach $(D^1 \times D^k)$ to each $\{\alpha_j\}_j$ and $\{\beta_j\}_j$. This is because $\partial(D^1 \times D^k) = \partial(S^1 \times D^k)$.

Therefore we now have the necessary tools for constructing $\mathscr{W}$.

Definition 104 (Constructing $\mathscr{W}$). Let X be a cellular complex with closed $(k, k+1)$ vector path V. Let $\dagger X$ be the vector path subdivision of X with respect to V. Then attach $(D^1 \times D^k)$ to each $\{\alpha_j\}_j$ and $\{\beta_j\}_j$.

Finally, attach $(D^2 \times D^k)$ to the sub-complex C where C is $\{\gamma_j\}_j$, $\{\alpha_j\}_j \cap \{\gamma_j\}_j \cap \{\beta_j\}_j$ and the newly attached $(D^1 \times D^k)$. This is possible since $C \cong S^2 \times D^k$ and $\partial(S^2 \times D^k) = \partial(D^2 \times D^k)$. We call the resulting complex $\mathscr{W}$.

7.2 Examples

Recall that in Example 80 we could perform surgery on S^2 to obtain $S^2 \sqcup S^2$. We then recall that by Theorem 86 that if we could perform surgery on a manifold to obtain another manifold then the spaces are cobordant. Therefore S^2 and $S^2 \sqcup S^2$ are cobordant. We also recall that by Example 19 a cube and sphere are homeomorphic. Therefore we will show a cube and a pair of cubes are locally cellular cobordant.

Example 105. Let M be the cube as described in Example 102. Let N be two copies of the cube described in Example 102. For this example we will take the sub complexes $X = M$, and $Y = N$.

Therefore consider $\dagger X$ in Figure 41. Then attach $(D^1 \times D^1)$ to each $\{\alpha_j\}_j$ and $\{\beta_j\}_j$. We see this in Figure 42.

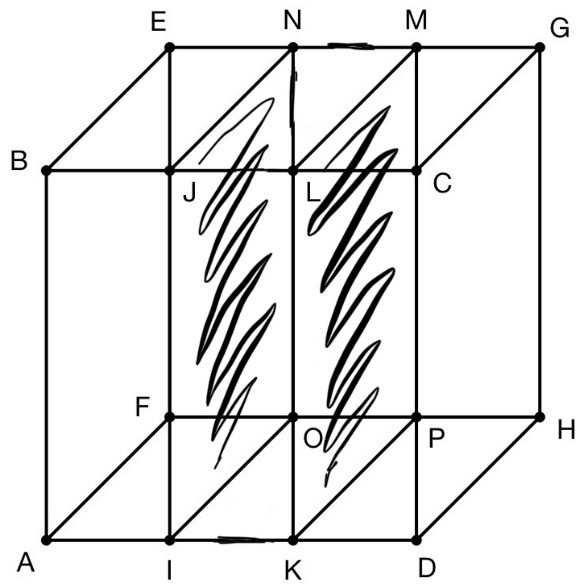

Figure 42: Cube with vector path subdivision and attached 2-cells.

Finally, attach $(D^2 \times D^1)$ to the sub-complex C where C is $\{\gamma_j\}_j$, $\{\alpha_j\}_j \cap \{\gamma_j\}_j \cap \{\beta_j\}_j$ and the newly attached $(D^1 \times D^1)$. We see this in Figure 43, where the yellow is the $(D^2 \times D^1)$.

Notice that in our figure we have a cube as well as a pair of cubes as sub-complexes of $\mathscr{W}$. Therefore we say a cube is locally cellular cobordant to a pair of cubes.

We have shown above that our cube constructed is locally cellular cobordant to a pair of our cubes. This is expected since surgery and cobordant is connected through Theorem 86. Therefore we wish to examine examples of cellular surgery and show we can create a cobordism between the original space and the resulting space of the cellular surgery.

We notice that in our definition of locally cellular cobordant we only need to focus

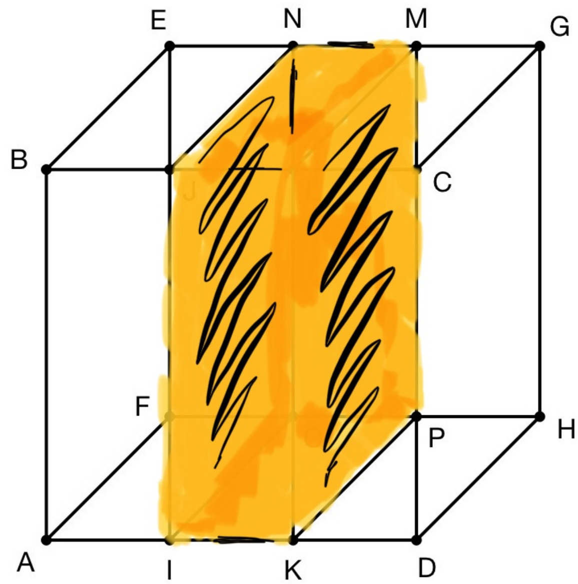

Figure 43: $\mathscr{W}$ for cube and two cubes

on the parts of the complexes that differ. That is if M is our complex before cellular surgery, and N is the complex after then we need only to analyze where M and N differ.

Example 106. Recall that in Examples 90, 92, 96, 97 and 98 the parts of our complex that differ are homeomorphic to a sphere and a pair of spheres. We have shown in the Example 105 that we can construct $\mathscr{W}$ for those spaces. Therefore, we say the complex we start with in each of those examples is locally cellular cobordant to the complexes resulting from cellular surgery, which is again what we would expect.

Example 107. Consider the torus as pictured in Figure 18. Now choose the same closed $(1,2)$-vector path on the torus pictured in Figure 19. Then we create the following vector path subdivision as we see in Figure 44.

Now attach $(D^1 \times D^1)$ to each $\{\alpha_j\}_j$ and $\{\beta_j\}_j$. Finally we attach $(D^2 \times D^1)$ to the sub-complex C where C is $\{\gamma_j\}_j$, $\{\alpha_j\}_j \cap \{\gamma_j\}_j \cap \{\beta_j\}_j$ and the newly attached $(D^1 \times D^1)$. We see this in figure 45 below.

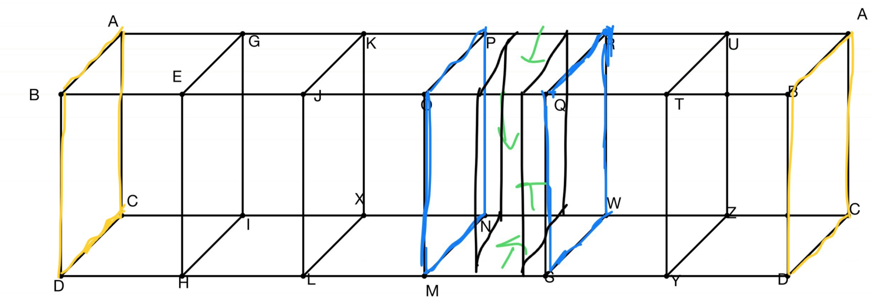

Figure 44: Torus with vector path subdivision

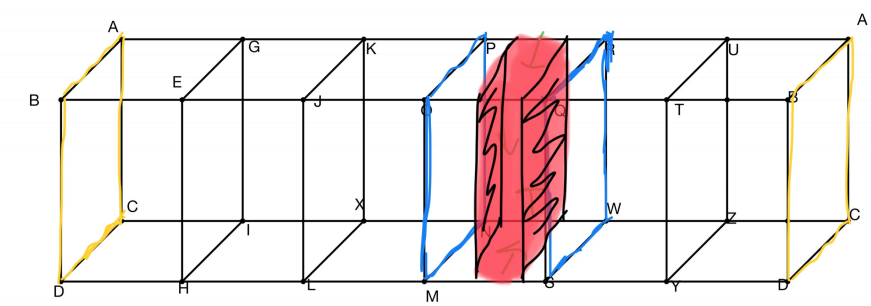

Figure 45: Cobordism for the torus and a cylinder.

We see that the above $\mathscr{W}$ in Figure 45 has both the torus from figure 18 and a cylinder as sub-complexes. Therefore we say a torus and a cylinder are locally cellular cobordant.

Example 108. Consider our torus with rooms in Figure 21. Choose the same closed $(1,2)$-vector path on the "torus with rooms as pictured" in Figure 26. Then we create the following vector path subdivision.

We attach $(D^1 \times D^1)$ to each $\{\alpha_j\}_j$ and $\{\beta_j\}_j$. We also attach $(D^2 \times D^1)$ to the sub-complex C where C is $\{\gamma_j\}_j$, $\{\alpha_j\}_j \cap \{\gamma_j\}_j \cap \{\beta_j\}_j$ and the newly attached $(D^1 \times D^1)$. However we cannot picture what this would look like. At first glance we might try to draw

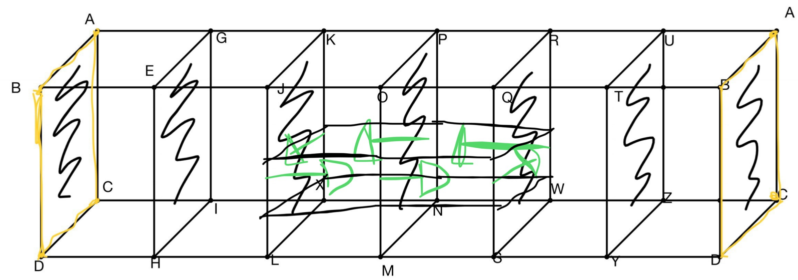

Figure 46: Torus with rooms vector path subdivision.

in the pieces on the complex above. However it would seem that $(D^1 \times D^1)$ and $(D^2 \times D^1)$ intersect the complex. However we need a fourth dimension to show this intersection is not actually occurring.

We notice that our $\mathscr{W}$ can quickly become difficult to construct. In our last example we need higher dimensions in order to accurately represent our new complex $\mathscr{W}$.

Also, we notice that our definition of cobordant is related to our generalization of surgery. Therefore, by this definition we have the exact same result as Theorem 87. The idea is to obtain one or the other we simply choose the same closed vector path. Therefore, this result is not interesting since it does not require discrete Morse theory as we hoped it would.

CHAPTER 8

Links to Smooth Case and Future Research

8.1 Future Research

As we noted at the end of Chapter 7, our generalization of cobordism and surgery were deliberately related to one another. We note that since the connection does not need discrete Morse theory to connect the definitions. Therefore, we would like to generalize our definition of cobordant, in a way that does not rely on the definition of surgery.

In the smooth case we noticed that surgery theory and cobordism are not immediately clear on their connection to one another. However, we found there was a connection between the two through smooth Morse theory. Therefore we want to find a way to continue our generalization of cobordism. As of now our definition relies heavily on our generalization of surgery. We would like to find a way to define our generalization of cobordism without using closed vector paths. Therefore, we would not have the immediate connection we have now. We want this generalization to allow for the possibility discrete Morse theory to provide the connection.

A theorem exists [6] stating that if a spin manifold of dimension at least 5 admits a metric with positive scalar curvature, performing surgery on that manifold preserves the positivity of the scalar curvature. The proof of this theorem relies heavily on the connection between cobordism and surgery. We are hoping this result or a form of this result holds for our generalization.

In order to further investigate this problem we must have find a consistent metric for curvature on a cellular complex. There have been attempts by Robin Forman, however none that help solve this specific problem.

We also would like to analyze how the topology is changing after performing our generalization of surgery on a cellular complex. We would like to know what invariants are preserved if any. Generally speaking we would like to analyze the results that came from smooth surgery, and analyze them if they hold for our generalization of surgery.

REFERENCES

[1] Armstrong, M. A. *Basic Topology.* Springer, 2011.

[2] Cappell, Sylvain, et al. *Surveys on Surgery Theory : Volume 2 Papers Dedicated to C.T.C. Wall.* (AM-149). Princeton University Press, 2014.

[3] Forman, R. (1998a). Morse theory for cell complexes, *Adv. Math.* **134**, 90-145.

[4] Forman, R. (1998b). Combinatorial vector fields and dynamical systems, *Math. Z.* **228** (1998), 629-681.

[5] Forman, R. (2002). A user guide to discrete Morse theory, *Sem. Lothar. Combin* **48**, Art. B48c, 35pp.

[6] Gromov, M. and Lawson, Jr, H.B.,(1980) The classification of simply connected manifolds of positive scalar curvature, *Ann. of Math.***112**, 423–434.

[7] Hatcher, A. (2001). *Algebraic topology.* Cambridge: Cambridge University Press.

[8] Knudson, K. P. (2015). Morse theory: smooth and discrete. Singapore: World Scientific Publishing Co. Pte. Ltd.

[9] Matsumoto, Y., and Hudson, K. (2002). An introduction to Morse theory. Providence, RI: American Math. Soc.

[10] Milnor, John W., and David W. Weaver. *Topology from the Differentiable Viewpoint.* Princeton University Press, 1997.

VITA

Ryan J. Gueli

EDUCATION

Master of Science student in Mathematics at Sam Houston State University, January 2016 – present. Thesis title: “A Generalization of Surgery to Cellular Complexes.”

Bachelor of Science (December 2015) in mathematics, University of Buffalo, Buffalo, New York.

ACADEMIC EMPLOYMENT

Graduate Teaching Assistant, Department of Mathematics and Statistics, Sam Houston State University, January 2016 - present. Responsible for delivering every lecture, and all other aspects of course management, such as, creating activities to monitor learning, grading and recording grades, suggesting solutions to assigned problems, addressing student performance issues, developing ways to improve learning and understanding, assigning final grades.

ACADEMIC AWARDS

Oustanding Graduate Student from the Office of Graduate Studies at Sam Houston State University 2017-2018

Outstanding Graduate Student from the Department of Mathematics at Sam Houston State University 2017-2018

Graduate Assistantship with Department of Mathematics and Statistics, Sam Houston State University

Member of Pi Mu Epsilon

President of Student chapter of AMS at Sam Houston State University from May 2016 - present.